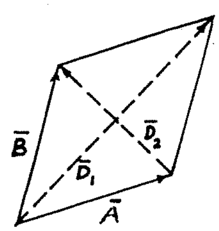
PROBLEM SET #1 SOLUTIONS

(1) P.2-5 Expand $\bar{A} \times (\bar{A} \times \bar{X}) = \bar{A}(\bar{A} \cdot \bar{X}) - \bar{X}(\bar{A} \cdot \bar{A})$,

or $\bar{A} \times \bar{B} = p\bar{A} - A^2 \bar{X}$.

$\therefore \quad \bar{X} = \frac{1}{A^2}(p\bar{A} + \bar{B} \times \bar{A})$.

(2) P.2-7



$\bar{D}_1 = \bar{B} + \bar{A}, \qquad \bar{D}_2 = \bar{B} - \bar{A}.$

$$\bar{D}_1 \cdot \bar{D}_2 = (\bar{B} + \bar{A}) \cdot (\bar{B} - \bar{A})$$
$$= \bar{B} \cdot \bar{B} - \bar{A} \cdot \bar{A} = 0$$

for a rhombus.

$\therefore \quad \bar{D}_1 \perp \bar{D}_2$.

(3) P.2-9 $\bar{a}_A = \bar{a}_x \cos\alpha + \bar{a}_y \sin\alpha$,
$\bar{a}_B = \bar{a}_x \cos\beta + \bar{a}_y \sin\beta$.

a) $\bar{a}_A \cdot \bar{a}_B = \cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$.

b)
$$\bar{a}_B \times \bar{a}_A = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix} = \bar{a}_z(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= \bar{a}_z \sin(\alpha - \beta).$$

$\therefore \quad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

(4) P. 2-20 $\bar{F}\cdot d\bar{\ell} = [\bar{a}_x xy + \bar{a}_y(3x-y^2)]\cdot(\bar{a}_x dx + \bar{a}_y dy)$
$= xy\,dx + (3x-y^2)\,dy.$

a) Along direct path ①. The equation of P_1P_2 is
$$y = \frac{3}{2}(x-1).$$

$$\underset{(\text{Path ①})}{\int_{P_1}^{P_2}} \bar{F}\cdot d\bar{\ell} = \underset{(\text{Path ①})}{\int_{P_1}^{P_2}} [xy\,dx + (3x-y^2)dy]$$
$$= \int_5^3 \frac{3}{2}x(x-1)dx + \int_6^3 (2y+3-y^2)dy$$
$$= -37 + 27 = -10.$$

b) Along path ②. This path has two straight-line segments. From P_1 to A: $x=5$, $dx=0$, $\bar{F}\cdot d\bar{\ell} = (15-y^2)dy$. From A to P_2: $y=3$, $dy=0$, $\bar{F}\cdot d\bar{\ell} = 3x\,dx$. Hence,
$$\underset{(\text{Path ②})}{\int_{P_1}^{P_2}} \bar{F}\cdot d\bar{\ell} = \int_6^3 (15-y^2)\,dy + \int_5^3 3x\,dx = 18 - 24 = -6.$$
$$\neq \underset{(\text{Path ①})}{\int_{P_1}^{P_2}} \bar{F}\cdot d\bar{\ell} \longrightarrow \text{Vector field } \bar{F} \text{ is not conservative.}$$

(5) P. 2-24 On the surface of the sphere, $R = 5$.
$$\oint_S (\bar{a}_R 3\sin\theta)\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi} (\bar{a}_R 3\sin\theta)\cdot(\bar{a}_R 5^2\sin\theta)\,d\theta\,d\phi$$
$$= \int_0^{2\pi}\int_0^{\pi} 75\sin^2\theta\,d\theta\,d\phi = 75\pi^2.$$

(6) P. 2-26 In spherical coordinates, $\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R)$, if $\bar{A} = \bar{a}_R A_R$.

a) $\bar{A} = f_1(\bar{R}) = \bar{a}_R R^n$, $A_R = R^n$.
$$\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^{n+2}) = (n+2)R^{n-1}$$

b) $\bar{A} = f_2(\bar{R}) = \bar{a}_R \frac{k}{R^2}$, $A_R = kR^{-2}$
$$\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(k) = 0.$$

(7) P. 2-32 a) $\bar{D} = \bar{a}_R \frac{\cos^2\phi}{R^3}$, $ds = R^2\sin\theta\,d\theta\,d\phi$.
$$\oint \bar{D}\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{2}-1\right)\sin\theta\,d\theta\,\cos^2\phi\,d\phi = -\pi.$$

b) $\bar{\nabla}\cdot\bar{D} = -\frac{\cos^2\phi}{R^4}$, $dv = R^2\sin\theta\,dR\,d\theta\,d\phi$.
$$\int_V \bar{\nabla}\cdot\bar{D}\,dv = \int_0^{2\pi}\int_0^{\pi}\int_1^2\left(-\frac{\cos^2\phi}{R^2}\right)\sin\theta\,dR\,d\theta\,d\phi = -\pi.$$

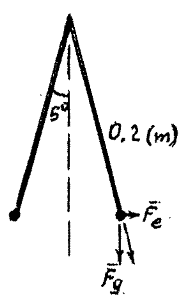
(8) P.3-5 $\overrightarrow{Q_1P} = -\bar{a}_x 2 - \bar{a}_y$; $\overrightarrow{Q_2P} = -\bar{a}_x 3 + \bar{a}_y$

$$\bar{E}_{P1} = \frac{Q_1}{4\pi\epsilon_0(\sqrt{5})^3}(-\bar{a}_x 2 - \bar{a}_y); \quad \bar{E}_{P2} = \frac{Q_2}{4\pi\epsilon_0(\sqrt{10})^3}(-\bar{a}_x 3 + \bar{a}_y).$$

a) No x-component of $\bar{E}_p$: $-\frac{2Q_1}{(\sqrt{5})^3} - \frac{3Q_2}{(\sqrt{10})^3} = 0$, or $\frac{Q_1}{Q_2} = -\frac{3}{4\sqrt{2}}$.

b) No y-component of $\bar{E}_p$: $-\frac{Q_1}{(\sqrt{5})^3} + \frac{Q_2}{(\sqrt{10})^3} = 0$, or $\frac{Q_1}{Q_2} = \frac{1}{2\sqrt{2}}$.

(9) P.3-6



At equilibrium, electric force $\bar{F}_e$ and gravitational force $\bar{F}_g$ must add to give a resultant along the thread.

$$\frac{F_e}{F_g} = \tan 5° = 0.0875,$$

$$F_g = mg = 9.80\times10^{-4}\ (N).$$

$$F_e = \frac{Q^2}{4\pi\epsilon_0(2\times0.2\sin 5°)^2} = 7.41\times10^{-12}Q^2\ (N).$$

$\longrightarrow Q = 3.40\ (nC)$.

(10) P.3-10 Use Gauss's law: $\oint_S \bar{E}\cdot d\bar{s} = Q/\epsilon_0$.

a) $\bar{E}$ is normal to the two faces at $x = \pm0.05$ (m), where $\bar{E} = \pm\bar{a}_x 5$ and $\bar{a}_n = \pm\bar{a}_x$ respectively.

$$Q = 2\epsilon_0(5\times0.1^2) = 0.1\epsilon_0 = 8.84\times10^{-13}\ (C)$$

b) $\bar{E} = \bar{a}_r(100x)\cos\phi - \bar{a}_\phi(100x)\sin\phi = \bar{a}_r(100r\cos^2\phi) - \bar{a}_\phi(100r\sin^2\phi)$.

$$\oint_S \bar{E}\cdot\bar{a}_n\, ds = \int_0^{2\pi}(100\times0.05\cos^2\phi)(0.1\times0.05)d\phi = 0.025\pi.$$

$$Q = 0.025\pi\epsilon_0 = 6.94\times10^{-13}\ (C).$$

(11) P.3-11 Spherical symmetry: $\bar{E} = \bar{a}_R E_R$. Apply Gauss's law.

1) $0 \le R \le b$. $4\pi R^2 E_{R1} = \frac{\rho_0}{\epsilon_0}\int_0^R\left(1-\frac{R^2}{b^2}\right)4\pi R^2 dR = \frac{4\pi\rho_0}{\epsilon_0}\left(\frac{R^3}{3} - \frac{R^5}{5b^2}\right)$,

$$E_{R1} = \frac{\rho_0}{\epsilon_0}R\left(\frac{1}{3} - \frac{R^2}{5b^2}\right).$$

2) $b \ge R < R_i$. $4\pi R^2 = \frac{\rho_0}{\epsilon_0}\int_0^b\left(1-\frac{R^2}{b^2}\right)4\pi R^2 dR = \frac{8\pi\rho_0}{15\epsilon_0}b^3$,

$$E_{R2} = \frac{2\rho_0 b^3}{15\epsilon_0 R^2}.$$

3) $R_i < R < R_o$. $E_{R3} = 0$.

4) $R > R_o$. $E_{R4} = \frac{2\rho_0 b^3}{15\epsilon_0 R^2}$.

(12) P.3-13 $W=-q\int \bar{E}\cdot d\bar{\ell} = -q\int (y\,dx + x\,dy)$.

a) Along the parabola $x=2y^2$, $dx = 4y\,dy$.

$$W = -q\int_1^2 6y^2\,dy = -14q = 28\ (\mu J).$$

b) Along the straight line $x=6y-4$, $dx = 6\,dy$.

$$W = -q\int_1^2 (12y-4)\,dy = -14q = 28\ (\mu J).$$

(13) Assume initial velocity of α-particle is V, mass m.

Kinetic energy is $\frac{1}{2}mV^2$

The distance of closest approach is when the potential energy $= \frac{1}{2}mV^2$

$$P.E = \frac{100e^2}{4\pi\varepsilon_0 R} \qquad e = \text{proton charge}$$

$$\text{So} \quad \frac{1}{2}mV^2 = \frac{100e^2}{4\pi\varepsilon_0 R}$$

$$R = \frac{50e^2}{\pi m\varepsilon_0 V^2}$$

(14) The volume charge density in the sphere is

$$\rho = \frac{Q}{\frac{4}{3}\pi a^3}$$

The charge in an annular shell at radius R is $\Delta Q = \dfrac{4\pi R^2 Q\, dR}{\frac{4}{3}\pi a^3} = 4\pi R^2 \rho\, dR$

The work required to bring 1 C up from infinity to this shell is

$$\Delta W = \frac{\Delta Q}{4\pi \varepsilon_0 R} = \frac{4\pi R^2 \rho\, dR}{4\pi \varepsilon_0 R} = \frac{R\rho\, dR}{\varepsilon_0}$$

Once you are inside the shell of charge no further work is required to move the 1C to (0,0,0). The work to bring the 1C to (0,0,0) against the whole sphere of charge is

$$W = \int_0^a \frac{R\rho\, dR}{\varepsilon_0} = \frac{a^2 \rho}{2\varepsilon_0} = \frac{a^2 Q}{\frac{8}{3}\pi a^3 \varepsilon_0} = \frac{3Q}{8\pi a \varepsilon_0}$$