ENEE 380 Problem Set #4

SOLUTIONS

(1) P.4-1 Use subscripts d and a to denote dielectric and air regions respectively. $\nabla^2 V = 0$ in both regions.

$$V_d = c_1 y + c_2,\quad \bar{E}_d = -\bar{a}_y c_1,\quad \bar{D}_d = -\bar{a}_y \epsilon_0 \epsilon_r c_1.$$

$$V_a = c_3 y + c_4,\quad \bar{E}_a = -\bar{a}_y c_3,\quad \bar{D}_a = -\bar{a}_y \epsilon_0 c_3.$$

B.C: At $y=0$, $V_d = 0$; at $y=d$, $V_a = V_0$;

at $y = 0.8d$: $V_d = V_a$, $\bar{D}_d = \bar{D}_a$.

Solving: $c_1 = \dfrac{V_0}{(0.8+0.2\epsilon_r)d}$, $c_2 = 0$, $c_3 = \dfrac{\epsilon_r V_0}{(0.8+0.2\epsilon_r)d}$, $c_4 = \dfrac{(1-\epsilon_r)V_0}{1+0.25\epsilon_r}$.

a) $V_d = \dfrac{5yV_0}{(4+\epsilon_r)d}$, $\bar{E}_d = -\bar{a}_y \dfrac{5V_0}{(4+\epsilon_r)d}$.

b) $V_a = \dfrac{5\epsilon_r y - 4(\epsilon_r - 1)d}{(4+\epsilon_r)d} V_0$, $\bar{E}_a = -\bar{a}_y \dfrac{5\epsilon_r V_0}{(4+\epsilon_r)d}$.

c) $(\rho_s)_{y=d} = -(D_a)_{y=d} = \dfrac{5\epsilon_0 \epsilon_r V_0}{(4+\epsilon_r)d}$.

$(\rho_s)_{y=0} \doteq (D_d)_{y=0} = -\dfrac{5\epsilon_0\epsilon_r V_0}{(4+\epsilon_r)d}$.

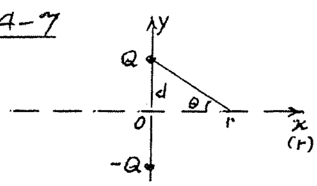
(2) P.4-3 At a point where V is a maximum (minimum) the second derivatives of V with respect to x, y, and z would all be negative (positive); their sum could not vanish, as required by Laplace's equation.

(3) P.4-6 Poisson's eq. $\nabla^2 V = -\dfrac{A}{\epsilon r} \longrightarrow \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial V}{\partial r}\right) = -\dfrac{A}{\epsilon r}$.

Solution: $V = -\dfrac{A}{\epsilon} r + c_1 \ln r + c_2$.

B.C.: $\begin{cases} \text{At } r=a, & V_0 = -\frac{A}{\epsilon}a + c_1 \ln a + c_2. \\ \text{At } r=b, & 0 = -\frac{A}{\epsilon}b + c_1 \ln b + c_2. \end{cases}$ $\quad c_1 = \dfrac{\frac{A}{\epsilon}(b-a) - V_0}{\ln(b/a)}$, $\quad c_2 = \dfrac{V_0 \ln b + \frac{A}{\epsilon}(a\ln b - b\ln a)}{\ln(b/a)}$.

(4) P.4-7



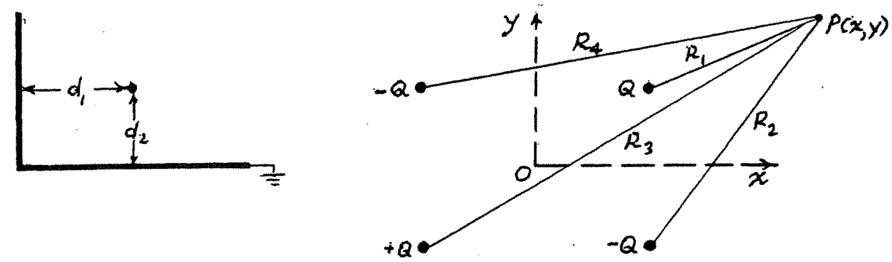
$$\bar{E}\Big|_{y=0} = -\bar{a}_y \frac{Q}{4\pi\epsilon R^2}(2\sin\theta) = -\bar{a}_y \frac{Qd}{2\pi\epsilon(d^2+r^2)^{3/2}}.$$

a) $\rho_s = \bar{a}_y \cdot \epsilon \bar{E}\Big|_{y=0} = -\dfrac{Qd}{2\pi(d^2+r^2)^{3/2}}$.

b) $\int_0^\infty \rho_s\, 2\pi r\, dr = -Q$.

(5)

P.4-8



Consider the conditions in the xy-plane (z=0).

a) $V_P = \frac{Q}{4\pi\epsilon}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right)$, where

$R_1 = [(x-d_1)^2 + (y-d_2)^2]^{1/2}$, $\quad R_2 = [(x-d_1)^2 + (y+d_2)^2]^{1/2}$,

$R_3 = [(x+d_1)^2 + (y+d_2)^2]^{1/2}$, $\quad R_4 = [(x+d_1)^2 + (y-d_2)^2]^{1/2}$.

$$\bar{E}_P = -\bar{\nabla} V_P = -\bar{a}_x \frac{\partial V_P}{\partial x} - \bar{a}_y \frac{\partial V_P}{\partial y}$$

$$= \bar{a}_x \frac{Q}{4\pi\epsilon}\left[-\frac{x-d_1}{R_1^3} + \frac{x-d_1}{R_2^3} - \frac{x+d_1}{R_3^3} + \frac{x+d_1}{R_4^3}\right]$$

$$+ \bar{a}_y \frac{Q}{4\pi\epsilon}\left[-\frac{y-d_2}{R_1^3} + \frac{y+d_2}{R_2^3} - \frac{y+d_2}{R_3^3} + \frac{y-d_2}{R_4^3}\right].$$

E_P will have a z-component if the point P does not lie in the xy-plane.

b) On the conducting half-planes, $\rho_s = D_n = \epsilon E_n$.
Along the x-axis, $y=0$: $R_1 = [(x-d_1)^2 + d_2^2]^{1/2} = R_2$,
and $R_3 = [(x+d_1)^2 + d_2^2]^{1/2} = R_4$.

$E_x = 0$, $E_y = \frac{Q}{2\pi\epsilon}\left[\frac{d_2}{R_1^3} - \frac{d_2}{R_3^3}\right]$.

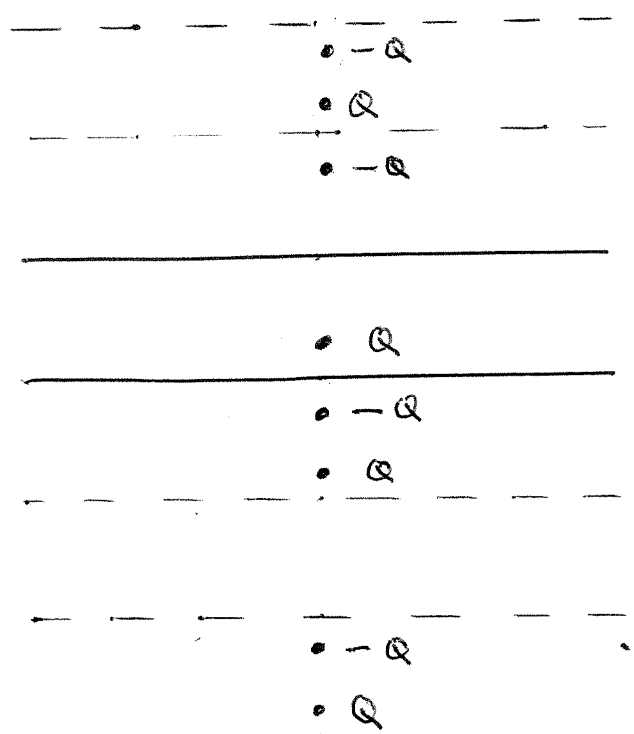
$$\therefore\ \rho_s(y=0) = \frac{Qd_2}{2\pi}\left\{\frac{1}{[(x-d_1)^2+d_2^2]^{3/2}} - \frac{1}{[(x+d_1)^2+d_2^2]^{3/2}}\right\}$$

$$= \begin{cases} 0, & \text{at } x=0. \\ \text{max.}, & \text{at } x = d_1. \end{cases}$$

Similarly for $\rho_s(x=0)$ on the vertical conducting conducting plane by changing x to y and $d_1 \leftrightarrow d_2$.

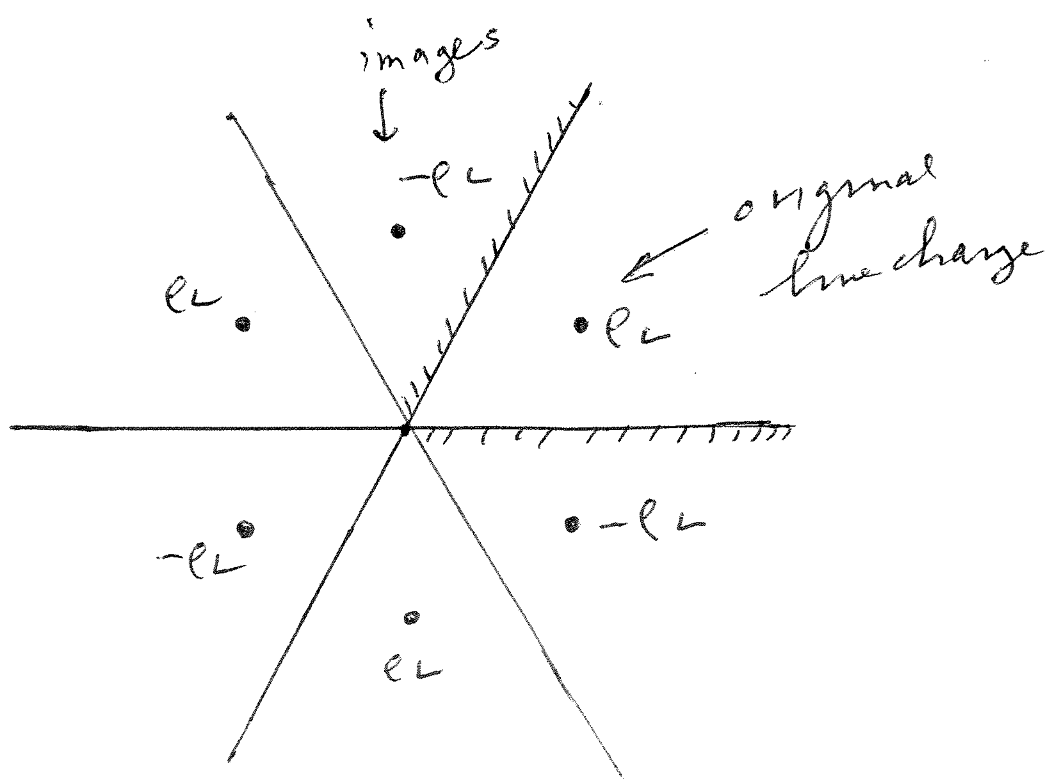
(6) (4.9) The required system of charges is as shown below.

(a)



and so on. There are an infinite number of image charges

(b)



(7) The capacitance required is C_{12} as shown below

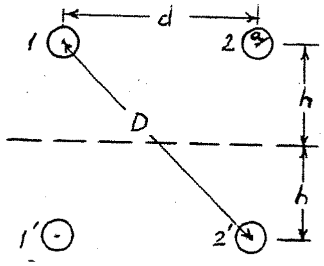


Image wires 1′ and 2′.

The potential differences referring to the ground are:

$$V_{10} = \frac{1}{2}V_{11'} = \frac{\rho_{\ell 1}}{2\pi\epsilon_0}\ln\frac{2h}{a} + \frac{\rho_{\ell 2}}{2\pi\epsilon_0}\ln\frac{D}{d}, \quad ①$$

$$V_{20} = \frac{1}{2}V_{22'} = \frac{\rho_{\ell 1}}{2\pi\epsilon_0}\ln\frac{D}{d} + \frac{\rho_{\ell 2}}{2\pi\epsilon_0}\ln\frac{2h}{a}, \quad ②$$

where $D = (4h^2+d^2)^{1/2}$

Eqs. ① and ② can be solved for $\rho_{\ell 1}$ and $\rho_{\ell 2}$ in terms of V_{10} and V_{20}:

$$\rho_{\ell 1} = \Delta_0\left(V_{10}\ln\frac{2h}{a} - V_{20}\ln\frac{D}{a}\right) = c_{11}V_{10} + c_{12}V_{20},$$

$$\rho_{\ell 2} = \Delta_0\left(-V_{10}\ln\frac{D}{a} + V_{20}\ln\frac{2h}{a}\right) = c_{21}V_{10} + c_{22}V_{20},$$

where $\Delta_0 = \dfrac{2\pi\epsilon_0}{(\ln\frac{2h}{a})^2 - (\ln\frac{D}{a})^2}$.

From Eqs. (3-157a, b, & c):

$$C_{12} = -c_{12} = \Delta_0 \ln\frac{D}{a} = \frac{2\pi\epsilon_0 \ln(D/a)}{(\ln\frac{2h}{a})^2 - (\ln\frac{D}{a})^2},$$

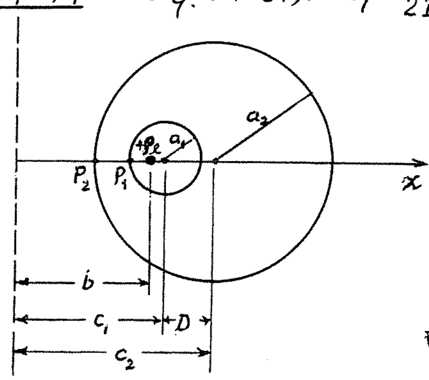
$$C_{10} = C_{20} = c_{11} + c_{12} = \frac{2\pi\epsilon_0}{\ln(2h/a) + \ln(D/a)}.$$

(8)

P. 4-14 Eq. (4-61): $c_1 = \frac{1}{2D}(a_2^2 - a_1^2 - D^2)$; Eq. (4-62): $c_2 = \frac{1}{2D}(a_2^2 - a_1^2 + D^2)$.

Eq. (4-55): $b^2 = c_1^2 - a_1^2$;

Eq. (4-56): $b^2 = c_2^2 - a_2^2$.



a) $V = \dfrac{\rho_\ell}{2\pi\epsilon_0}\ln\dfrac{r_2}{r_1}$ ← r_2: Distance to $-\rho_\ell$; r_1: Distance to $+\rho_\ell$.

At P_1: $r_2 = b + (c_1 - a_1)$, $r_1 = b - (c_1 - a_1)$.

At P_2: $r_2 = b + (c_2 - a_2)$, $r_1 = b - (c_2 - a_2)$.

$$V_1 - V_2 = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\left[\frac{b+(c_1-a_1)}{b-(c_1-a_1)}\,\frac{b-(c_2-a_2)}{b+(c_2-a_2)}\right].$$

Expressing b, c_1 & c_2 in terms of D, a_1 & a_2 and simplifying:

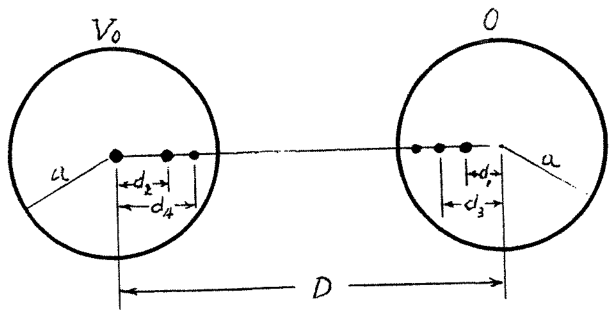
$$V_1 - V_2 = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\left\{\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right) + \left[\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right)^2 - 1\right]^{1/2}\right\}$$

$$C' = \frac{\rho_\ell}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\ln\left\{\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right) + \left[\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right)^2 - 1\right]^{1/2}\right\}} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{a_1^2+a_2^2-D^2}{2a_1a_2}\right)} \quad (F/m).$$

b) Force per unit length $F' = \dfrac{\rho_\ell^2}{2\pi\epsilon_0(4b^2)} = \dfrac{D^2\rho_\ell^2}{2\pi\epsilon_0[(a_1^2+a_2^2-D^2)^2 - 4a_1^2D^2]}$ (N/m).

(9)

P.4-16



a)

Q_0 and system of image charges:

In left sphere

Q_0 at $d_0 = 0$.

$Q_2 = \frac{a^2}{D(D-d_1)} Q_0$ at d_2.

$Q_4 = \frac{a^4}{D(D-d_1)(D-d_2)(D-d_3)} Q_0$ at d_4.

⋮

$Q_{Ln} = Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}}$ at d_n. $(n = 2, 4, 6, \cdots)$

In right sphere

$-Q_1 = -\frac{a}{D} Q_0$ at d_1.

$-Q_3 = -\frac{a}{D-d_2} Q_2 = -\frac{a^3}{D(D-d_1)(D-d_2)} Q_0$ at d_3.

⋮

$-Q_{Rn} = -Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}}$ at d_n. $(n = 1, 3, 5, \cdots)$

$d_m = \frac{a^2}{D-d_{m-1}}$, $m = 1, 2, 3, \cdots$; $d_0 = 0$.

b) $C = \frac{Q_0 + \sum_n Q_{Ln}}{V_0} = 4\pi\epsilon_0 a \left[1 + \sum_{n=2,4,\cdots} \left(\prod_{m=1}^{n} \frac{a}{D-d_{m-1}} \right) \right]$.

(10) Image Charge Problem

Choose some reasonable parameters

$a := 10^{-4} \cdot 5$ sphere radius

$d := 2 \cdot 10^{-3}$ d/2 is distance between center of charged sphere and ground plane

$q0 := 10^{-9}$ Charge on sphere in Coulombs

Choose charge on sphere to be equivalent to charge q0=q at (0,0,0)

First image charge is q1=-q at point P1 (x1,0,0)

$x1 := d$

$q1 := -q0$

Second image charge is q2 at point P2 (x2,0,0)

$$x2 := \frac{a^2}{d}$$

$$q2 := q0 \cdot \frac{a}{d}$$

Third image charge is q3=-q2 at point P3 (x3,0,0)

$$x3 := d - \frac{a^2}{d}$$

$q3 := -q2$

Fourth image charge is q4 at point (x4,0,0)

$$x4 := \frac{a^2}{x3}$$

$$q4 := -q3 \cdot \frac{a}{x3}$$

Fifth image charge is q5 at (x5,0,0)

$x5 := d - x4$

$q5 := -q4$

Sixth image charge is q6 at (x6,0,0)

$x6 := \frac{a^2}{x5}$

$q6 := -q5 \cdot \frac{a}{x5}$

Make a grid of points

$i := 1, 2 .. 200$

$j := 1, 2 .. 200$

$x_i := (i - 1) \cdot \frac{d}{398} - 0.000001$ $\qquad x_{101} = 5.015 \times 10^{-4}$

$y_j := (j - 1) \cdot \frac{d}{398} + 0.00001$

$r0_{i,j} := \sqrt{(x_i)^2 + (y_j)^2}$ distance from origin to grid points

$r1_{i,j} := \sqrt{(x_i - x1)^2 + (y_j)^2}$ Distances from first image charge to grid points

$r2_{i,j} := \sqrt{(x_i - x2)^2 + (y_j)^2}$

$r3_{i,j} := \sqrt{(x_i - x3)^2 + (y_j)^2}$

$r4_{i,j} := \sqrt{(x_i - x4)^2 + (y_j)^2}$ Distances from image charges to grid points

$r5_{i,j} := \sqrt{(x_i - x5)^2 + (y_j)^2}$

$r6_{i,j} := \sqrt{(x_i - x6)^2 + (y_j)^2}$

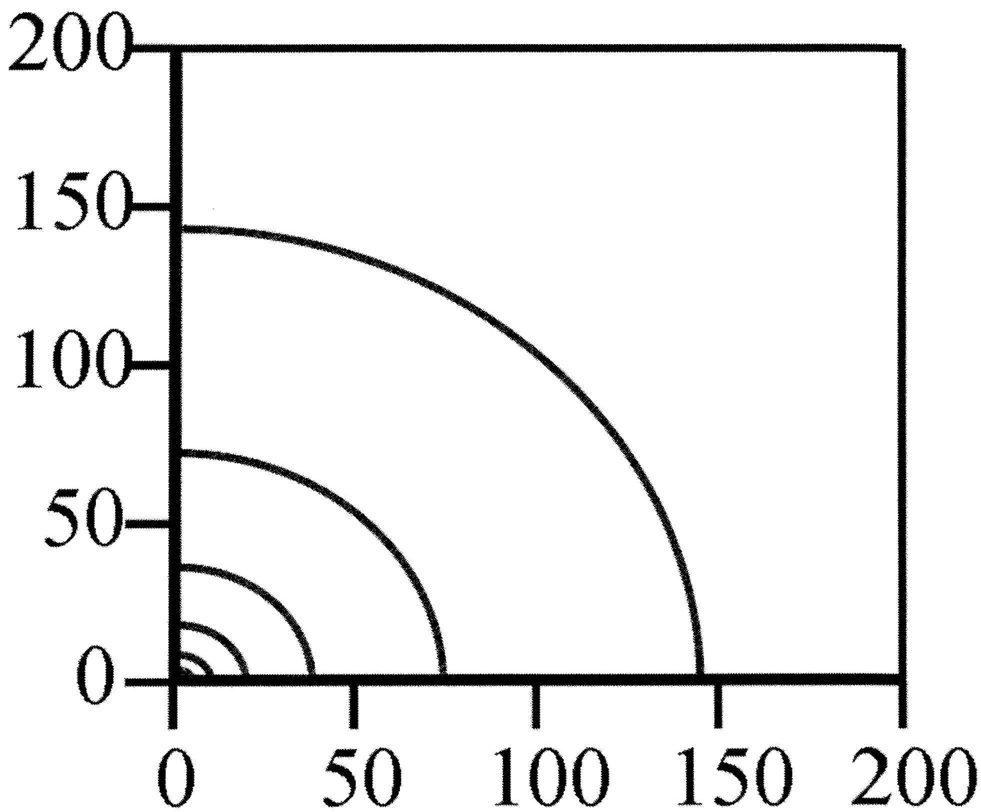
Compute potentials at grid points, forget about $1/4\pi\varepsilon_0$ normalization factor

$$V0_{i,j} := \frac{q0}{r0_{i,j}} \qquad V1_{i,j} := \frac{q1}{r1_{i,j}} \qquad V2_{i,j} := \frac{q2}{r2_{i,j}} \qquad V3_{i,j} := \frac{q3}{r3_{i,j}}$$

$$V4_{i,j} := \frac{q4}{r4_{i,j}} \qquad V5_{i,j} := \frac{q5}{r5_{i,j}} \qquad V6_{i,j} := \frac{q6}{r6_{i,j}}$$

Plot equipotentials with different potentials included

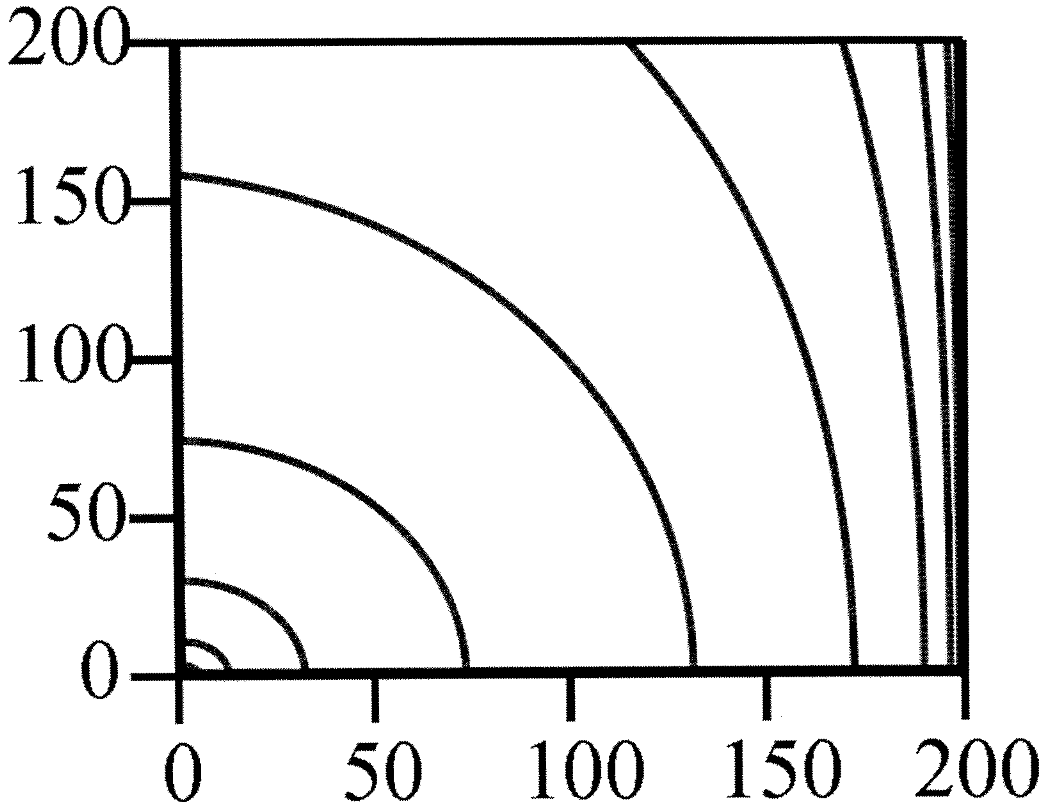
$$V_{i,j} := \log\left(V0_{i,j}\right)$$



V

Circularly symmetric contour lines
for initial charged sphere.
Note that equipotentials do not match
conducting plane. The sphere actually
lies along the 101 contour line.
In this figure sphere center is at
lower left. Ground plane is at right.

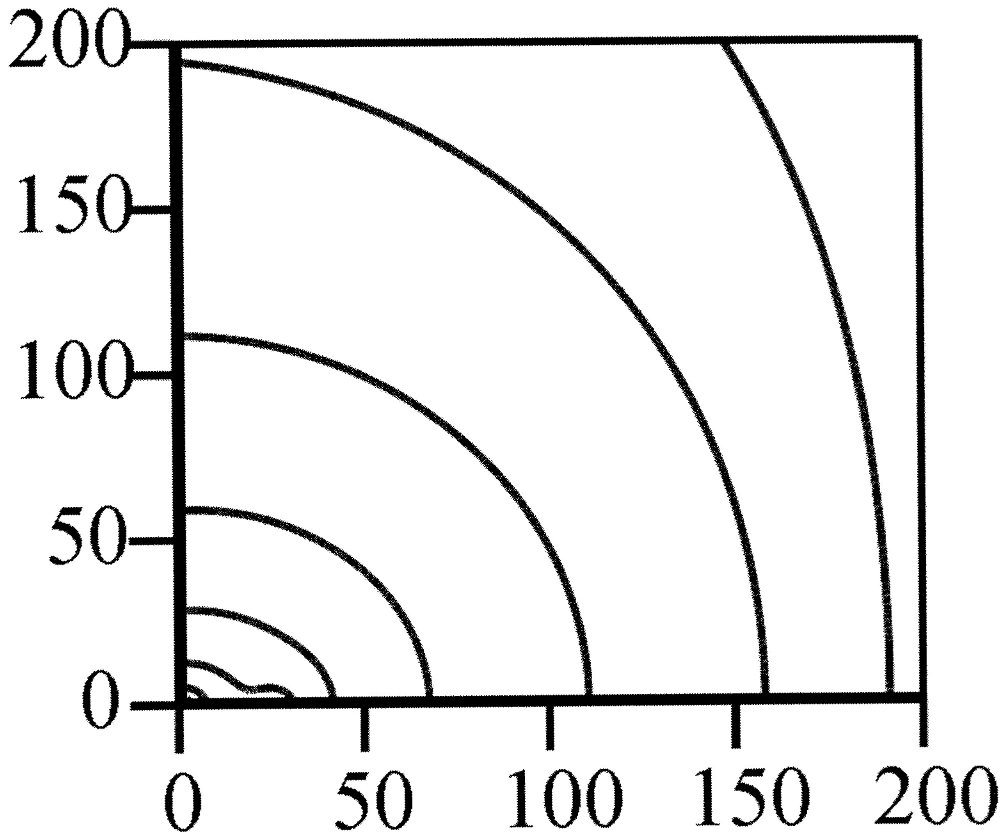
$V_{i,j} := \log\left(V0_{i,j} + V1_{i,j}\right)$ 2 charges only Use log to get better looking plots



V

Contours for two charges, note that
equipotential matches ground plane on right.
The sphere lies along the 101 counter line.
The first 4 contour lines are inside the sphere

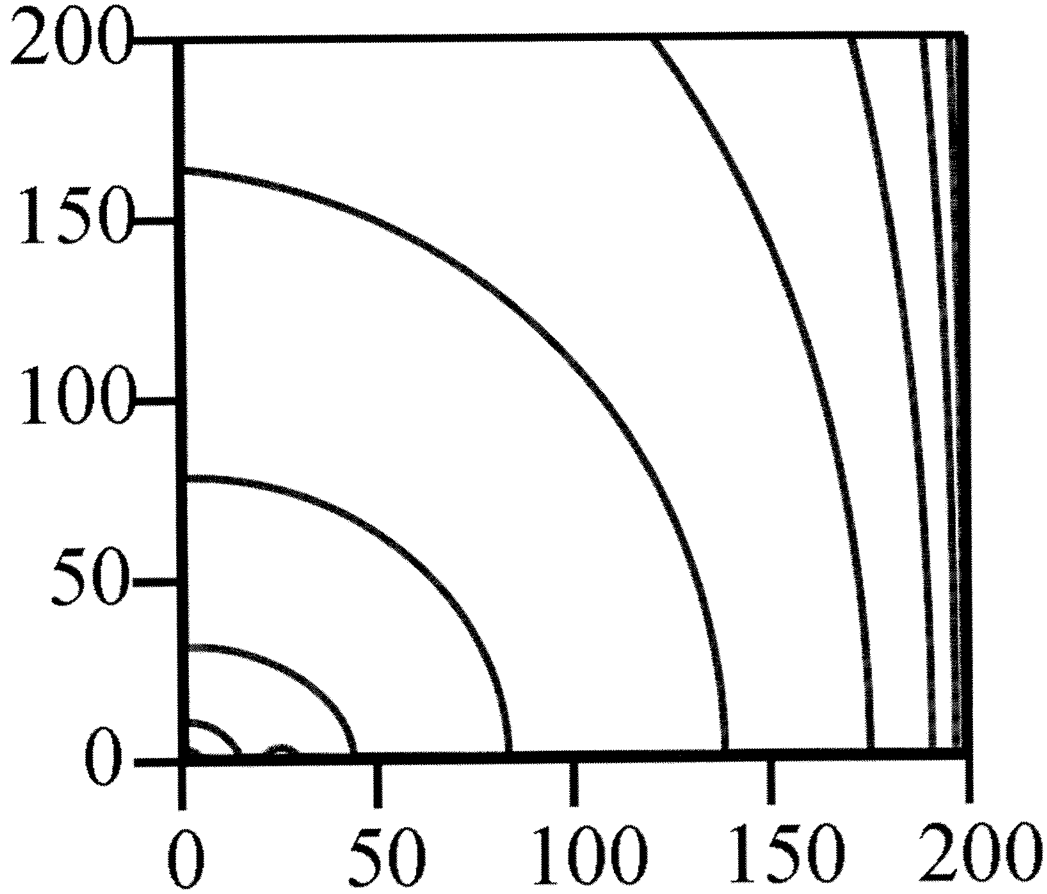
$$V_{i,j} := \log\left(V0_{i,j} + V1_{i,j} + V2_{i,j}\right)$$



V

Contours for three charges. These match on sphere, but not on ground plane.

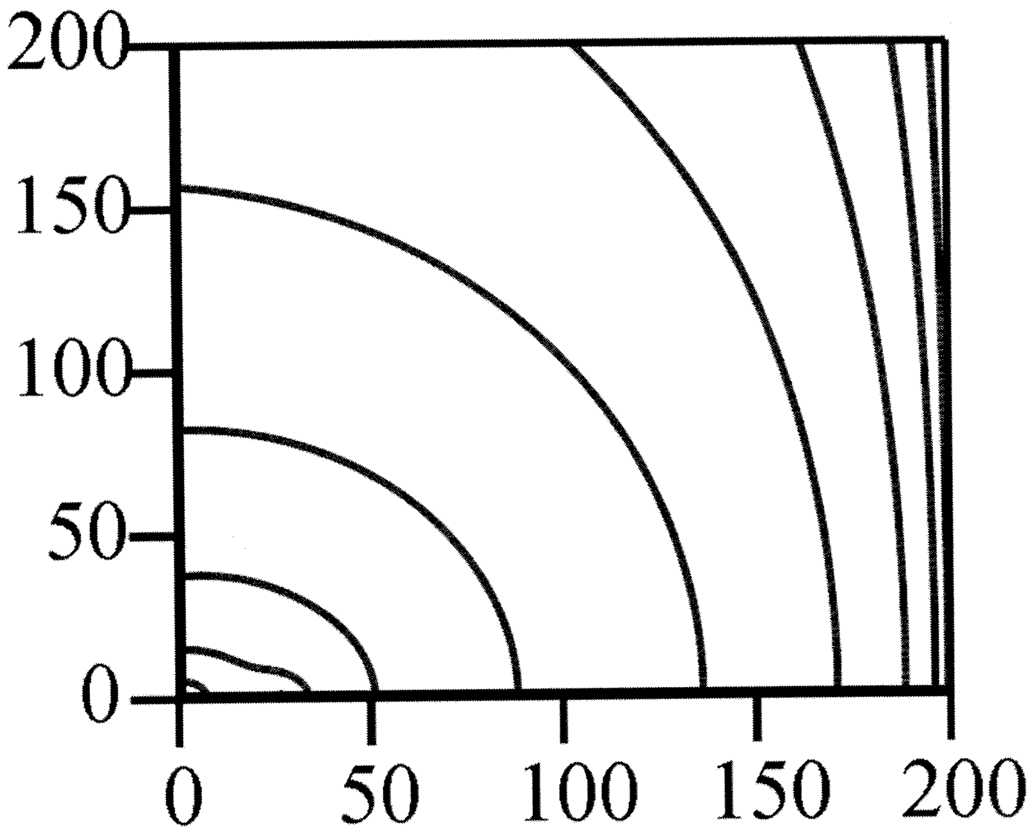
$V_{i,j} := \log\left(V0_{i,j} + V1_{i,j} + V2_{i,j} + V3_{i,j}\right)$



V

Contours for 4 charges

$$V_{i,j} := \log\left(V0_{i,j} + V1_{i,j} + V2_{i,j} + V3_{i,j} + V4_{i,j} + V5_{i,j} + V6_{i,j}\right)$$



V

Contour for 6 charges. Note that equipotentials match on sphere and ground plane. Sphere actually lies along 101 contour line.