ENEE 380. Homework #5

4/17/03 due 4/24/03

(1) Cheng Problem (4.17)

(2) Cheng Problem (4.21)

(3) Cheng Problem (4.23)

(4) Cheng Problem (4.24)

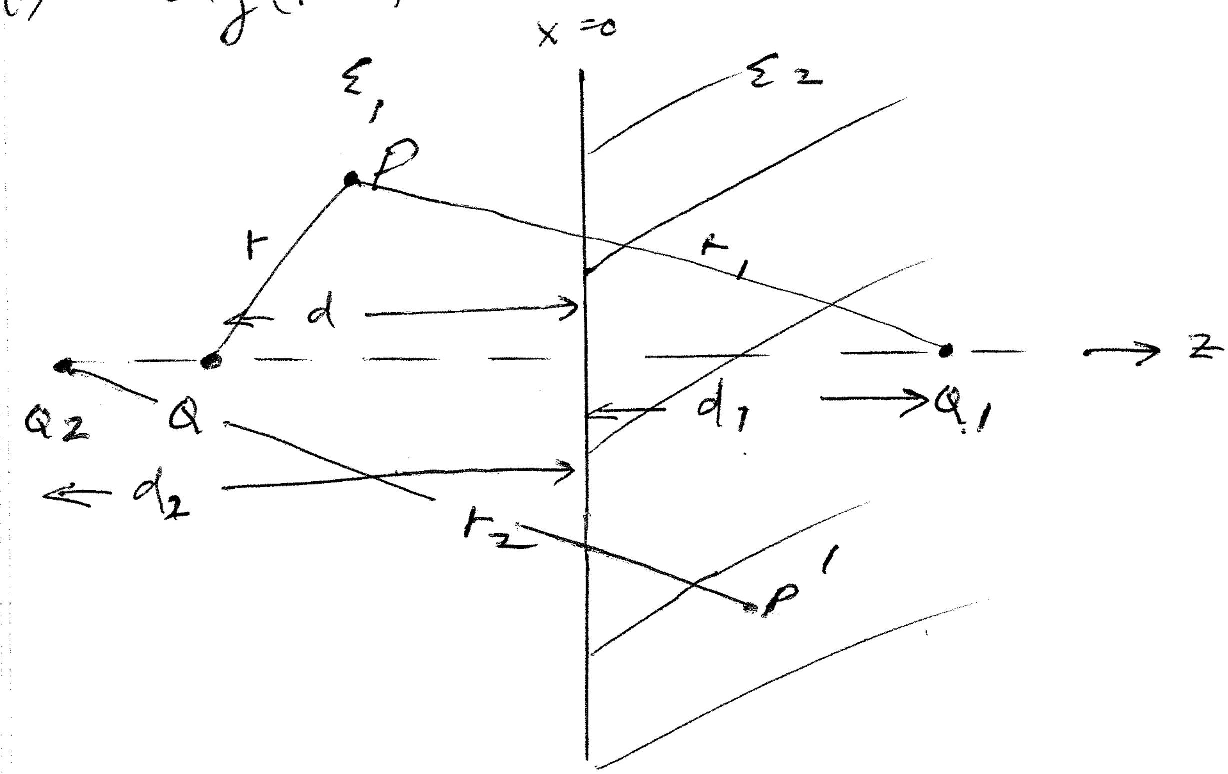
(5) Cheng Problem (4.27)

(6) Cheng Problem (4.29)(Challenge Question)

(7) A point charge of 1nC is placed 10mm above an infinite planar conductor. Produce a 3-D plot of the charge density on the conductor as a function of position.

ENEE 380 PROBLEM SET # 5
SOLUTIONS

(1) Cheng (4.17)



Charge Q produces a surface charge on the dielectric, which affects the field outside. We represent this effect by an image charge Q_1. The field inside is produced by charge Q_2. (We will discover that Q_2 & Q are actually coincident, and $d_1 = d$)

The potential at point P (in the image situation) is

$$V_p = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{Q_1}{r_1}\right)$$

The diel. const. does not enter because

we have replaced the dielectrics by image charges

Inside the dielectric the potential at a point like P' is

$$V_{p'} = \frac{Q_2}{4\pi\varepsilon_0 r_2}$$

To avoid an infinite field at the boundary $V_p' = V_p$ for points P & P' that coincide on the boundary. This can only happen for all points on the boundary if

$$r = r_1 = r_2$$

So Q_2 & Q coincide and $d_1 = d$
In addition $Q + Q_1 = Q_2$
This also ensures that tangential fields are continuous on the boundary

The second boundary condition is that the normal $\underline{D}$ must be continuous (since there is no free charge on the boundary). This requires $\varepsilon_1 \frac{\partial V_p}{\partial z} = \varepsilon_2 \frac{\partial V_{p'}}{\partial z}$

on the boundary $z = 0$

At an arbitrary point (x, y, z)
$r_2 = \left[x^2 + y^2 + (a + z)^2\right]^{1/2}$, $r_1 = \left[x^2 + y^2 + (a - z)^2\right]^{1/2}$

so

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = \frac{\partial}{\partial z}\left(\frac{1}{r_2}\right) = -\frac{(a+z)}{r^3}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r_1}\right) = \frac{a-z}{r_1^3}$$

For $z=0$ this make the boundary condition

$$\frac{\varepsilon_1 a (Q - Q_1)}{4\pi\varepsilon_0 r^3} = \frac{\varepsilon_2 a Q_2}{4\pi\varepsilon_0 r^3}$$

Therefore

$$Q + Q_1 = Q_2 \quad \text{and} \quad \varepsilon_1(Q - Q_1) = \varepsilon_2 Q_2$$

Giving finally

$$Q_1 = \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right) Q$$

$$Q_2 = \left(\frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2}\right) Q$$

(2) P. 4-21 $V(x,y) = \sum_n \sin\frac{n\pi}{a}x\left[A_n \sinh\frac{n\pi}{a}y + B_n\cosh\frac{n\pi}{a}y\right].$

At $y=0$, $V(x,0) = V_2 = \sum_n B_n \sin\frac{n\pi}{a}x \longrightarrow B_n = \begin{cases} \frac{4V_2}{n\pi}, & n=\text{odd}. \\ 0, & n=\text{even}. \end{cases}$

At $y=b$, $V(x,b) = V_1 = \sum_n \sin\frac{n\pi}{a}x\left[A_n \sinh\frac{n\pi}{a}b + B_n\cosh\frac{n\pi}{a}b\right]$

$\longrightarrow A_n \sinh\frac{n\pi}{a}b + B_n\cosh\frac{n\pi}{a}b = \begin{cases} \frac{4V_1}{n\pi}, & n=\text{odd}. \\ 0, & n=\text{even}. \end{cases}$

$\therefore A_n = \begin{cases} \frac{4}{n\pi \sinh(n\pi b/a)}\left(V_1 - V_2\cosh\frac{n\pi}{a}b\right), & n=\text{odd}. \\ 0, & n=\text{even}. \end{cases}$

(3) P. 4-23 Solution: $V(\phi) = A_0\phi + B_0.$

a) B.C. ①: $V(0) = 0 \longrightarrow B_0 = 0.$
B.C. ②: $V(\alpha) = V_0 = A_0\alpha \longrightarrow A_0 = \frac{V_0}{\alpha}.$
$\therefore V(\phi) = \frac{V_0}{\alpha}\phi, \quad 0 \le \phi \le \alpha.$

b) B.C. ①: $V(\alpha) = V_0 = A_1\alpha + B_1$
B.C. ②: $V(2\pi) = 0 = 2\pi A_1 + B$
$\longrightarrow A_1 = -\frac{V_0}{2\pi-\alpha}, \quad B_1 = \frac{2\pi V_0}{2\pi-\alpha}.$

$\therefore V(\phi) = \frac{V_0}{2\pi-\alpha}(2\pi-\phi), \quad \alpha \le \phi \le 2\pi.$

(4) P. 4-24 The solution is the superposition of that for Example 4-9 and that for Fig. 4-19 rotated 90° in the clockwise direction. (In both cases V_0 should be replaced by $V_0/2$.)

Inside: $V(r,\phi) = \frac{2V_0}{\pi}\sum_{n=\text{odd}}\frac{1}{n}\left(\frac{r}{b}\right)^n\left[\sin n\phi + \sin n\left(\phi+\frac{\pi}{2}\right)\right], \quad r<b.$

Outside: $V(r,\phi) = \frac{2V_0}{\pi}\sum_{n=\text{odd}}\frac{1}{n}\left(\frac{b}{r}\right)^n\left[\sin n\phi + \sin n\left(\phi+\frac{\pi}{2}\right)\right], \quad r>b.$

(5) P.4-27 V and $\bar{E}$ depend only on θ. $\longrightarrow$ Eq.(4-9): $\frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right)=0$.

a) Solution: $\frac{dV}{d\theta} = \frac{C_1}{\sin\theta} \longrightarrow V(\theta) = C_1 \ln\left(\tan\frac{\theta}{2}\right) + C_2$.

B.C. ① $V(\alpha) = V_0 = C_1 \ln\left(\tan\frac{\theta}{2}\right) + C_2$

② $V(\frac{\pi}{2}) = 0 = C_1 \ln\left(\tan\frac{\pi}{4}\right) + C_2 \longrightarrow C_2 = 0$

$\longrightarrow C_1 = \frac{V_0}{\ln[\tan(\alpha/2)]} \longrightarrow V(\theta) = \frac{V_0 \ln[\tan(\theta/2)]}{\ln[\tan(\alpha/2)]}$.

b) $\bar{E} = -\bar{a}_\theta \frac{dV}{R\,d\theta} = -\bar{a}_\theta \frac{V_0}{R \ln[\tan(\alpha/2)] \sin\theta}$.

c) On the cone: $\theta = \alpha$, $\rho_s = \epsilon_0 E(\alpha) = -\frac{\epsilon_0 V_0}{R \ln[\tan(\alpha/2)] \sin\theta}$.

On the grounded plane: $\theta = \pi/2$, $\rho_s = -\epsilon_0 E\left(\frac{\pi}{2}\right) = \frac{\epsilon_0 V_0}{R \ln[\tan(\alpha/2)]}$.

(6) P.4-29 $R \le b$: $V_i(R,\theta) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta)$; $R \ge b$, $V_o(R,\theta) = \sum_{n=0}^{\infty} \left(B_n R^n + C_n R^{-(n+1)}\right) P_n(\cos\theta)$.

For $R \gg b$, $V_o(R,\theta) = -E_0 z = -E_0 R\cos\theta \longrightarrow B_1 = -E_0$; $B_n = C_n = 0$ for $n \neq 1$.

$\therefore V_o(R,\theta) = -E_0 R\cos\theta + C_1 R^{-2}\cos\theta$.

$$\left.\begin{aligned} &\text{B.C. ① } V_i(b,\theta) = V_o(b,\theta) \longrightarrow A_1 b = -E_0 b + C_1 b^{-2} \\ &\text{② } \epsilon_r \left.\frac{\partial V_i}{\partial R}\right|_{R=b} = \left.\frac{\partial V_o}{\partial R}\right|_{R=b} \longrightarrow \epsilon_r A_1 = -E_0 - 2C_1 b^{-3} \end{aligned}\right\} \quad A_1 = -\frac{3E_0}{\epsilon_r+2}, \quad C_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2} E_0 b^3.$$

$V_i(R,\theta) = -\frac{3E_0}{\epsilon_r+2} R\cos\theta$, $\quad V_o(R,\theta) = -E_0 R\cos\theta + \frac{(\epsilon_r-1)b^3}{(\epsilon_r+2)R^2} E_0 \cos\theta$.

$\bar{E}_i(R,\theta) = -\bar{\nabla} V_i = \frac{3E_0}{\epsilon_r+2}(\bar{a}_R \cos\theta - \bar{a}_\theta \sin\theta) = \bar{a}_z \frac{3E_0}{\epsilon_r+2}$.

$\bar{E}_o(R,\theta) = -\bar{\nabla} V_o = \bar{a}_R \left[1 + \frac{2(\epsilon_r-1)b^3}{(\epsilon_r+2)R^3}\right] E_0 \cos\theta - \bar{a}_\theta \left[1 - \frac{(\epsilon_r-1)b^3}{(\epsilon_r+2)R^3}\right] E_0 \sin\theta$.

(7) $\varepsilon_0 := 8.854\cdot 10^{-12}$

$q := 10^{-9}$ location (0,0,10mm)

Take conductor to be in z=0 plane

Choose a grid to use to make plots

$x := -5\cdot 10^{-2}, 10^{-4} .. 5\cdot 10^{-2}$

$y := -5\cdot 10^{-2}, 10^{-4} .. 5\cdot 10^{-2}$

At the point(x,y,0) the potential from the main charge and its image is

$V(x,y) = \dfrac{q}{4\cdot\pi\cdot\varepsilon_0\cdot R} - \dfrac{q}{4\cdot\pi\cdot\varepsilon_0\cdot R}$ which is clearly zero

Calculate potential just above conductor at point (x,y,Δz)

$\Delta z := 10^{-9}$ small distance

$R_1(x,y) := \sqrt{x^2 + y^2 + \left(10^{-2} - \Delta z\right)^2}$ distances from each charge to the measurement point

$R_2(x,y) := \sqrt{x^2 + y^2 + \left(10^{-2} + \Delta z\right)^2}$

$V(x,y) := \dfrac{q}{4\cdot\pi\cdot\varepsilon_0\cdot R_1(x,y)} - \dfrac{q}{4\cdot\pi\cdot\varepsilon_0\cdot R_2(x,y)}$ Potential from main charge and image

$V(0,0) = 1.798\times 10^{-4}$

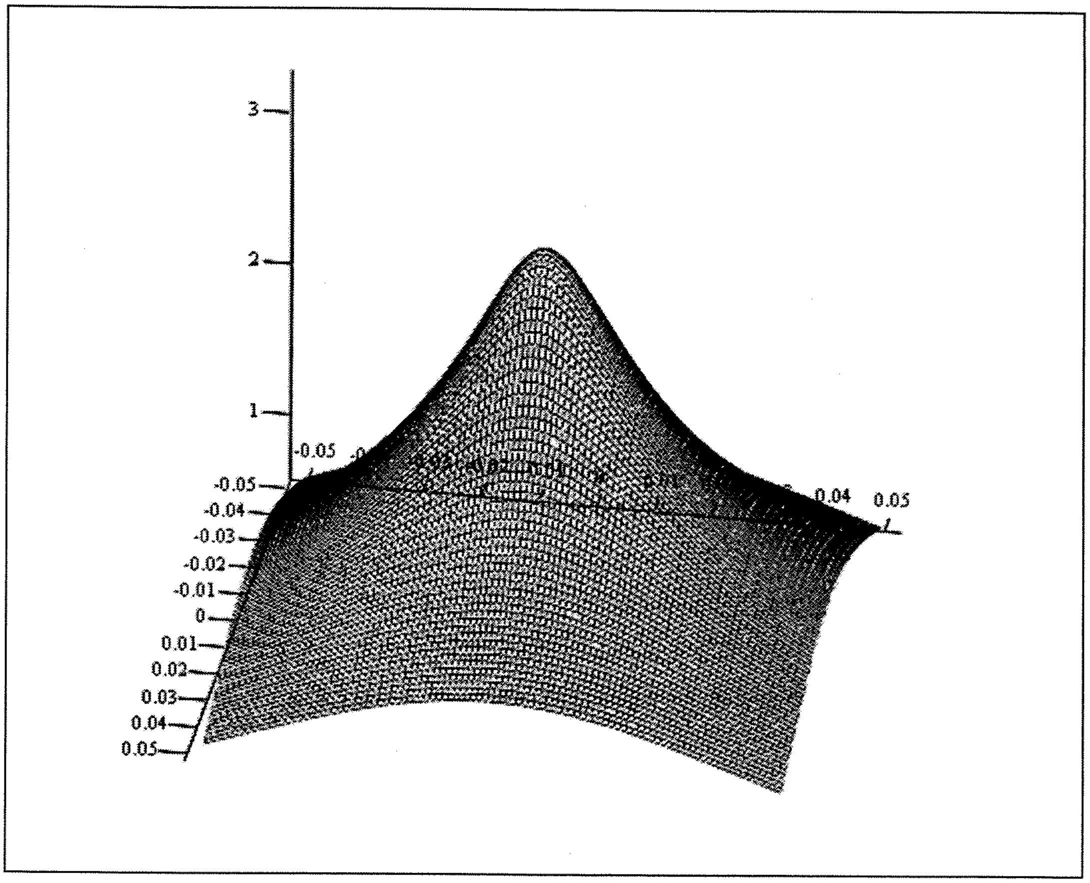
Field is the -gradient of potential, so

$E_n(x,y) := \dfrac{-(V(x,y))}{\Delta z}$

$\rho_s(x,y) := \varepsilon_0\cdot E_n(x,y)\cdot 10^9$ Charge density in nC/m^2

Plot log of charge density to get better looking plots

$\log\rho(x,y) := \log\left(-\rho_s(x,y)\right)$



logρ

Negative surface charge density plotted on a log scale