ENEE 380 Second Examination
May 6, 2003, 12:30pm - 1:45pm
ANSWER 3 QUESTIONS
If more than 3 are answered, best 3 will count

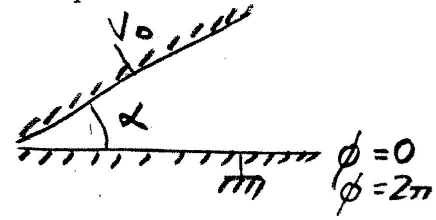
(1) From Gauss's law and the definition of the electrostatic potential V prove that

$$\nabla^2 V = -\frac{\rho}{\epsilon_r \epsilon_0} \quad \text{3pts.}$$

In the following geometry, which is infinite in the z-direction, calculate the potential for $0 < \phi < \alpha$ and $\alpha < \phi < 2\pi$. 7pts.

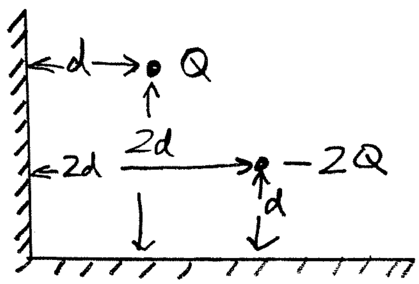
In cylindrical coordinates

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}.$$

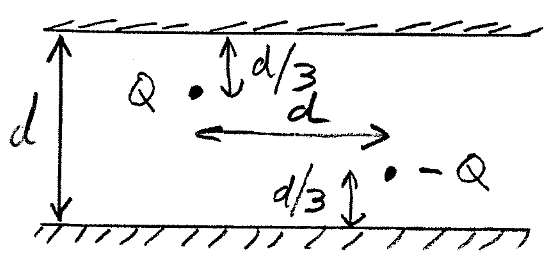


(2) What combination of images would you use to solve the following problems involving charges and infinite conducting planes.

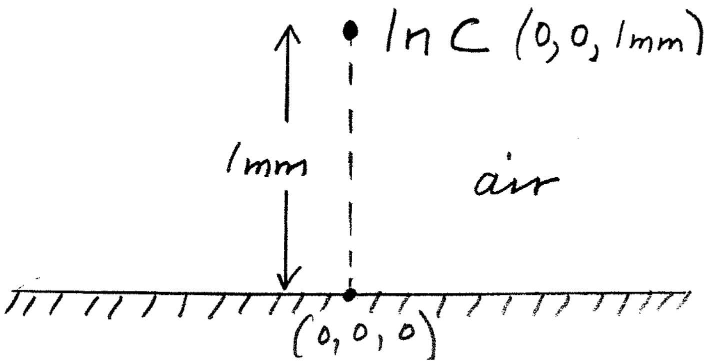
(a) (3pts.)



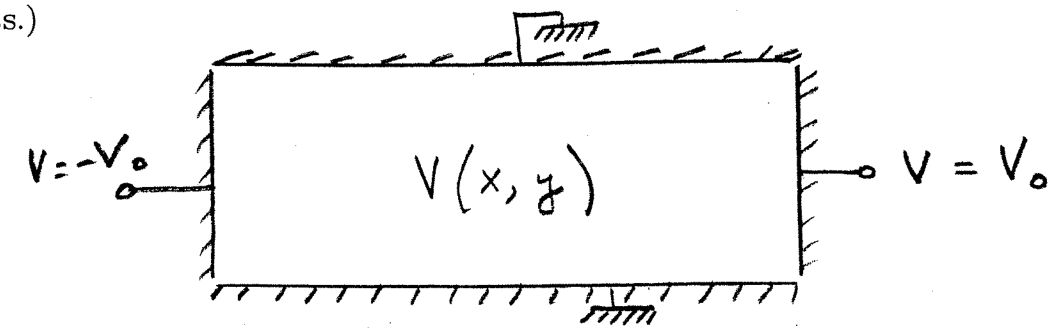
(b) (3pts.)



In the following arrangement, what is the charge density on the conductor at (0,0,0)? (4pts.)



(3) Explain the general form of the solution for the following electrostatic potential problem involving infinite conducting planes. You do not need to solve the problem completely. (7pts.)



In Cartesian coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

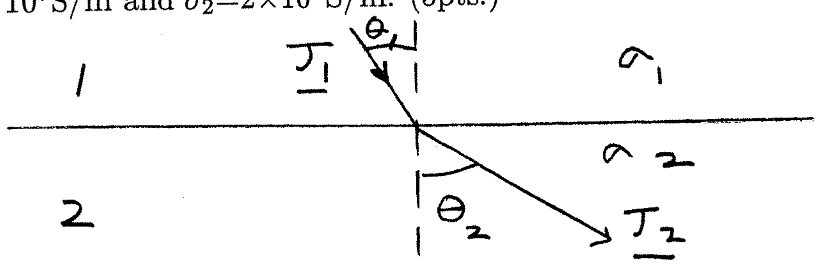
Explain how the current density in a geometric problem where the conductivity is the same everywhere can be found from the solution to Laplace's equation. (3pts.)

(4) Do two of the following:

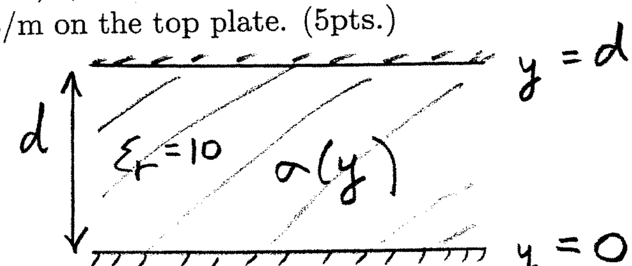
(a) Prove that the capacitance between two objects and the resistance between them are related by

$$RC = \frac{C}{G} = \frac{\epsilon_r \epsilon_0}{\sigma}. \quad \text{5pts.}$$

(b) In the following geometry involving a current density crossing a boundary between two different metals of different conductivities calculate $|\mathbf{J}_2|$ and θ_2. Take $|\mathbf{J}_1|$=1A/m^{-2}, θ_1=30°, $\sigma_1 = 10^7$S/m and σ_2=2×10^7S/m. (5pts.)



(c) Calculate the resistance of the following leaky capacitor. Take the plate spacing d=10mm, plate area 100mm^2, ϵ_r=10. The conductivity varies from linearly from 1 S/m on the bottom plate to 10S/m on the top plate. (5pts.)



$$\epsilon_0 = 8.854 \times 10^{-12} \quad \text{F/m}$$

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{Henry/m.}$$

ENEE 380, Fall 2002. Second Examination Solutions

(1) From Gauss's law

$$\mathrm{div}\mathbf{D} = \epsilon_r \epsilon_0 \mathrm{div}\mathbf{E} = \rho.$$

From the definition of the electrostatic potential $\mathbf{E} = \mathrm{grad}V$. Combining

$$\epsilon_r \epsilon_0 \mathrm{div\ grad}V = -\rho,$$

which gives the final result

$$\nabla^2 V = -\frac{\rho}{\epsilon_r \epsilon_0}.$$

The symmetry of the problem is such that there is only ϕ variation so

$$\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0.$$

We can exclude $r = 0$ and write this as

$$\frac{\partial^2 V}{\partial \phi^2} = 0,$$

which has the solution

$$V = A\phi + B.$$

For the region $0 < \phi < \alpha$ the boundary conditions $V(0) = 0$ and $V(\alpha) = V_0$ gives.

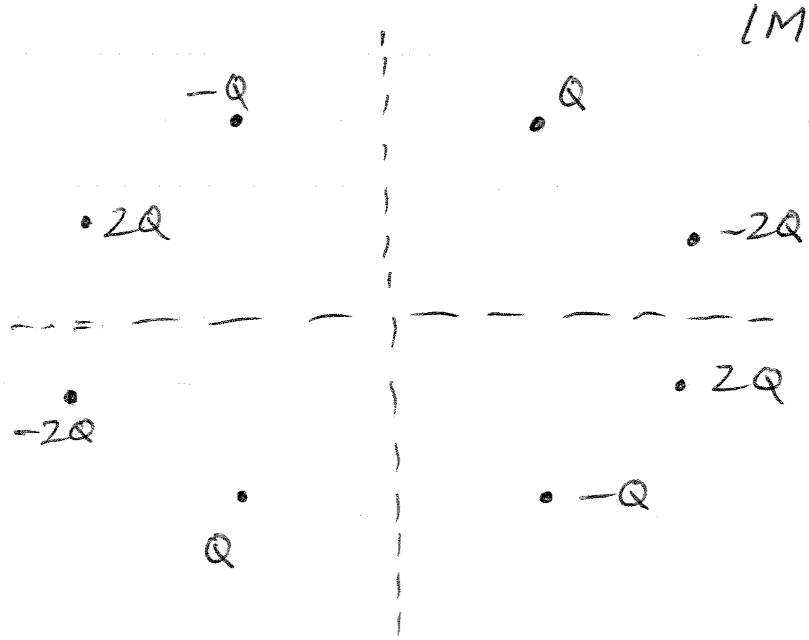
$$V = V_0(\frac{\phi}{\alpha}).$$

For the region $\alpha < \phi < 2\pi$ the boundary conditions are $V(\alpha) = V_0$ and $V(2\pi) = 0$, which gives

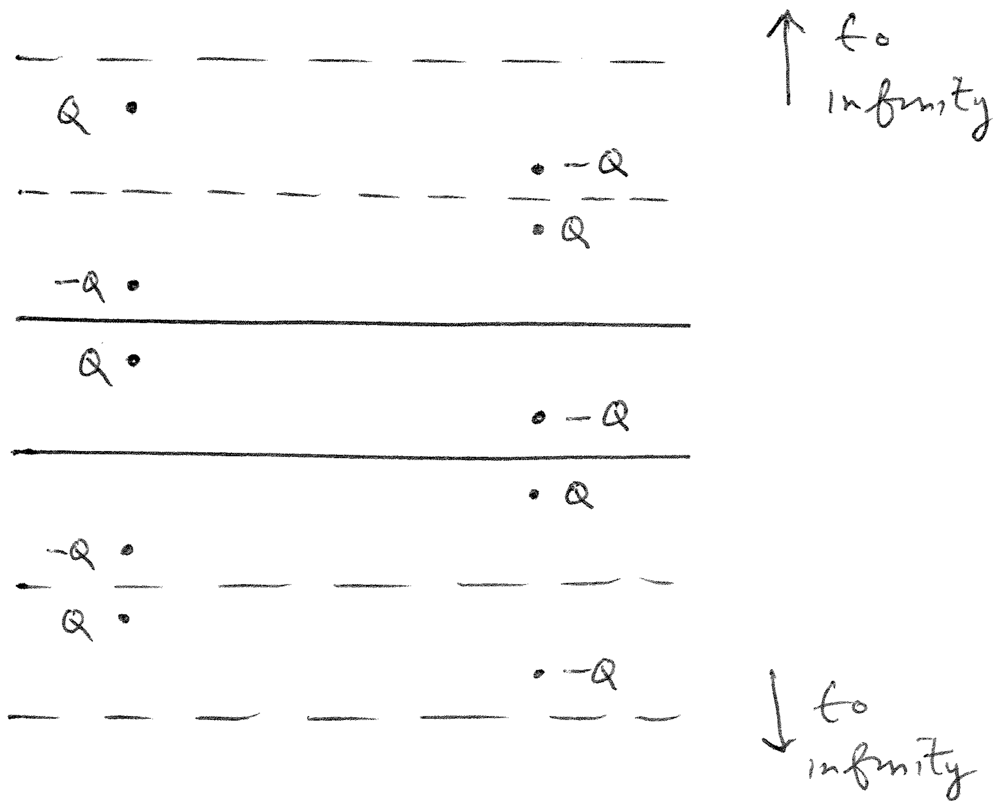
$$V = V_0 \frac{\phi - \alpha}{\alpha - 2\pi} + V_0.$$

(2)The arrangement of image charges for parts (a) and (b) are shown on a following page.

(2a) IMAGES



(2b)



(c) The field at (0,0,0) comes from the main 1nC charge at 1mm above the $z = 0$ plane and an image charge of -1nC at (0,0,-1mm).

$$E_z = -\frac{2 \times 10^{-9}}{4\pi\epsilon_0 \times 10^{-6}} = \frac{1.592 \times 10^{-4}}{\epsilon_0}.$$

The charge density is $\rho_s = \epsilon_0 E_z$, which gives $\rho_s = -1.592 \times 10^{-4}\mathrm{C/m^2}$.

(3) Choose the left and right hand walls to be at $x = -a$, $x = a$, respectively. Choose the top and bottom walls as $y = 0$ and $y = b$. The problem is now symmetrical about $x = 0$, and the plane x=0 is clearly an equipotential with $V = 0$. Each half of the problem is now like the problem with three walls at ground potential and the fourth wall at V_0 (or $-V_0$). The $Y(y)$ solution looks like $\sin ky$ in the vertical direction, where $kb == n\pi$ giving $k = n\pi/b$. The solution for $0 \leq x \leq a$ is

$$X(x) = A\mathrm{sinh}(kx) + B\mathrm{cosh}(kx).$$

The boundary conditions require

$$A\mathrm{sinh}(0) + B\mathrm{cosh}(0) = 0 \quad \text{and}$$

$$A\mathrm{sinh}(ka) + B\mathrm{cosh}(ka) = V_0,$$

which gives $B = 0$ and $A = V_0/\mathrm{sinh}(ka)$. The overall solution will be like (for $x > a$)

$$X(x)Y(y) = \sum_1^{\infty} C_n \left(\frac{\sinh(kx)\sin(ky)}{\sinh(ka)}\right).$$

It is easy to write a similar solution if the left and right-hand walls are taken as $x = 0$ and $x = 2a$. Only the sinh function is needed because $\mathrm{sinh}(-x) = -\mathrm{sinh}(x)$.

To show how current distribution can be found from Laplace's equation note that since curl $\mathbf{J} = 0$ (Ampère's Law) we can write $\mathbf{J} = -\mathrm{grad}\psi$. From

$$\mathrm{div}\mathbf{J} = -\frac{\partial \rho}{\partial t},$$

which gives div$\mathbf{J} = 0$ for no net charge changes with time, then

$$\mathrm{div\ grad}\psi = \nabla^2\psi = 0.$$

And $\mathbf{J} = \sigma \mathbf{E}$ so $\mathbf{J}$ and $\mathbf{E}$ are closely related.

(4) (a) The capacitance between two objects is

$$C = \frac{Q}{V} = \frac{\oint \epsilon_r \epsilon_0 \mathbf{E}.d\mathbf{S}}{-\int_1^2 \mathbf{E}.d\mathbf{l}}. \qquad (i)$$

The resistance is

$$R = \frac{V}{I} = \frac{-\int_1^2 \mathbf{E}.d\mathbf{l}}{\oint \mathbf{J}.\mathbf{dS}},$$

which is

$$R = \frac{V}{I} = \frac{-\int_1^2 \mathbf{E}.d\mathbf{l}}{\sigma \oint \mathbf{E}.d\mathbf{S}}. \qquad (ii)$$

Compare (i) and (ii)

$$RC = \frac{C}{G} = \frac{\epsilon_r \epsilon_0}{\sigma}.$$

(b) Normal $\mathbf{J}$ is continuous, so

$$J_1 \cos(\theta_1) = J_2 \cos(\theta_2), \qquad (iii)$$

where I have written J_1 and J_2 for the current density magnitudes. Tangential E-fields are continuous, so

$$E_{t_1} = E_{t_2},$$

which gives

$$\frac{J_1 \sin(\theta_1)}{\sigma_1} = \frac{J_2 \sin(\theta_2)}{\sigma_2}. \qquad (iv)$$

Solving (iii) and (iv) with $\theta_1 = 30°$,$\sigma_1 = 10^7$ and $\sigma_2 = 2 \times 10^7$ gives θ_2=49.107° and J_2=1.3229A/m^2.

(c) The current is $I = J_n A$, since normal current density is continuous. The resistance of a thin slab of the dielectric of thickness dy at y is

$$dR = \frac{dy}{A\sigma(y)}.$$

Clearly if σ varies linearly with y from 1 at $y = 0$ to 10 at $y = d$ then $\sigma(y) = 1 + 9y/d$. The overall resistance from bottom to top is

$$R = \int_0^d \frac{dy}{A(1 + 9y/d)},$$

which gives

$$R = \frac{d \ln(10)}{9A}.$$

Plugging in the numbers R=25.58ohm.