ENEE 380 ELECTROMAGNETIC THEORY

Second Examination Solutions

(1) To prove that $E_{t_1}=E_{t_2}$ construct a loop that runs parallel to the boundary on each side of the boundary. Let the long sides be of length w, and the short sides (perpendicular to the boundary) be of negligible length. Around the loop

$$\oint \mathbf{E} \cdot \mathbf{dl} = 0,$$

which gives $E_{t_1}w - E_{t_2}w = 0$, which gives the result desired.

To prove that $D_{n_1} - D_{n_2} = \rho_s$ make a Gaussian surface in the boundary that is a thin flat box of area 1 m^2 along the two sides of the boundary. Apply Gauss's theorem to this box

$$\oint \mathbf{D} \cdot \mathbf{dS} = Q = \rho_s.$$

On the large 1m^2 faces D_n is parallel to $\mathbf{dS}$ so the integral gives $D_{n_1} - D_{n_2} = \rho_s$, since there is no contribution to the surface integral over the small sides of the box that cut through the boundary.

Outside the dielectric sphere

$$\mathbf{E} = E_r = \frac{Q}{4\pi\epsilon_0 r^2} \quad \text{and}$$

$$V = \frac{Q}{4\pi\epsilon_0 r}.$$

Ar r=2 this gives E_r=2.25×10^9 V/m, V=4.49×10^9 V. At r=1 $V = V_1$=8.988×10^9 V. Inside the dielectric sphere

$$\mathbf{E} = E_r = \frac{Q}{4\pi\epsilon_0\epsilon_r r^2},$$

which gives E_r at r=0.5 = 3.595×10^{8}V/m. The potential difference between r=0.5 and r=1 is

$$V_{diff} = -\int_1^{0.5} \frac{Q}{4\pi\epsilon_0\epsilon_r r^2} = \frac{Q}{4\pi\epsilon_0\epsilon_r}.$$

This gives $V(r = 0.5) - V(r = 1)$=8.988×10^{7}V. Therefore, $V(r = 0.5) = 8.988 \times 10^7 + 8.988 \times 10^9 = 9.08 \times 10^9$V.

(2)Let the charge per unit length on the inner conductor be Q. On the outer conductor it is $-Q$. The field between the two conductors is found by applying Gauss's theorem to a cylindrical surface of length 1m around the inner conductor.

$$2\pi D_r = Q, \quad \text{which gives} \quad E_r = \frac{Q}{2\pi\epsilon_0\epsilon_r}.$$

$$\text{Applying} \quad V_{ab} = -\int_b^a E_r dr \quad \text{gives}$$

$$V_{ab} = \frac{Q \ln(b/a)}{2\pi\epsilon_0\epsilon_r}, \quad \text{and}$$

$$C = \frac{Q}{V_{ab}} = \frac{2\pi\epsilon_0\epsilon_r}{\ln(b/a)}.$$

With a=1mm, b=3mm, ϵ_r=10 C=5.05×10^{-10}F. The stored energy per unit length is

$$W_e = \tfrac{1}{2}CV^2, \quad \text{which gives} \quad W_e = 2.525 \times 10^{-10} \quad \text{J/m}.$$

(3) Following the solution to problem (2) above, the field near a line charge is

$$E_r = \frac{\rho_l}{2\pi\epsilon_0}.$$

Consequently the potential difference between two points distant r and r_0 from the line charge is

$$V = -\int_{r_0}^{r} E_r dr = \frac{\rho_l}{2\pi\epsilon_0} \ln(r_0/r).$$

If the line charge ρ_l and the axis of the cylindrical conductor are separated by d,then the work done in bringing up a 1C charge to a point M on the cylindrical conductor spaced r_i from the $-\rho_l$ line charge is

$$W = \frac{\rho_l}{2\pi\epsilon_0} \ln(r_0/r) - \frac{\rho_l}{2\pi\epsilon_0} \ln(r_0/r_i),$$

where r is the distance from point M to the ρ_l line charge, which gives

$$W = \frac{\rho_l}{2\pi\epsilon_0} \ln(r_i/r).$$

W must be the same for all points M on the cylindrical conductor: the surface of the conductor is an equipotential. Consequently, (r_i/r)=constant. By similar tringles $(r_i/r) = (a/d)$ and $(d_i/a) = (a/d)$. Consequently, $d_i = a^2/d$, which gives d_i=0.01mm.

(4) The image charge distibution that solves the problem is an arrangment of 7 additional charges, alternating + and - arranged symmetrically so that the 8 charges (Q plus the 7 image charges) are arranged in an 8 sided arrangement. For example there is a -Q image charge at (x,-y,0), a +Q image charge at (-x,-y,0) etc. Because the charge distribution has charges of the same sign facing each other on lines drawn through the origin, there is no net field at (0,0,0). Consequently, the induced charge density there is 0. The charge distribution along the bottom conductor starts at 0 at (0,0,0) decreases to a maximum negative value more or less opposite the charge Q, and then gradually rises to zero as you go along the x-axis. The behavior on the 45° conductor is similar. The total induced charge on both conductors together is -Q.