ENEE 381, Spring 2004. Homework Set 1

Due February 12, 2004

(1) Cheng Problem 6-15

(2) Cheng Problem 6-17

(3) Cheng Problem 6-18

(4) A circular coil of radius 100mm with 100 turns rotates about a vertical diameter at an angular velocity of 100 rad/s in a uniform horizontal magnetic flux of 0.01T. The resistance of the coil is 5ohm. Calulate the current induced in the coil and the energy dissipated in it. A small compass needle, which is free to move, is mounted at the center of the rotating coil. Calculate the angle with respect to the external flux direction at which it comes to equilibrium.

(5) A single electron is moving in the z-direction with velocity 10^6 m/s along the line $x = y$=1mm. The electron crosses the plane z=0 at $t = 0$. Plot the variation in electric field amplitude at the point (0,0,0) from t=-10ns to t=10ns.

Now, suppose that an infinite continuous line of electrons is moving in the same way, with a uniform spacing between electrons of 1mm. Show the variation in electric field from t=-10ns to t=10ns. What is the magnetic field **H** at the origin (0,0,0)?

(6) A circular metal disk of diameter 500mm rotates at 1000 rpm about a vertical axis in a vertical magnetic flux of 0.1T. See for example Cheng Example 7-3. What is the voltage generated beteen the axis and perimeter of the disk along a radius?

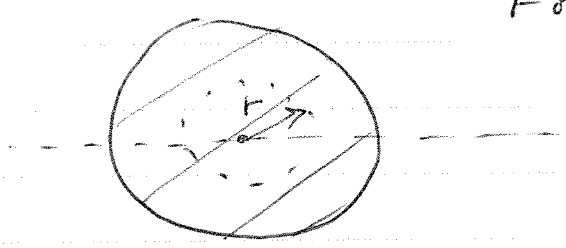
(7) The current in a region of space $\rho \leq$5mm is $\mathbf{J} = 10e^{-10^5 r^2}\hat{\mathbf{z}}$. Use the magnetic potential vector to give expressions for the magnetic field everywhere.

(8) A positive charge q moving with velocity $\mathbf{v} = v\hat{\mathbf{k}}$ enters a region of space where there is a uniform magnetic flux $\mathbf{B}=B_0\hat{\mathbf{j}}$ and a uniform electric field $\mathbf{E}=E_0\hat{\mathbf{i}}$. Write down the equations of motion of the charge and calculate its trajectory.

ENEE 381 SPRING 2004

Homework # 1

(1) Cheng 6-15



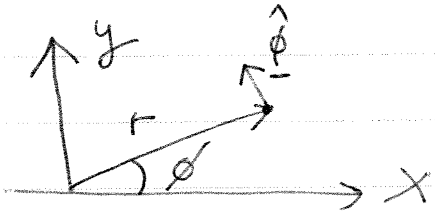
For initial solid conductor

At radius r inside conductor

$$\oint \underline{H} \cdot d\underline{l} = J \int_0^r 2\pi r' \, dr'$$

This gives $2\pi r H_\phi = \pi r^2 J$

$$H_\phi = \frac{rJ}{2}$$



Convert to $(x\ y)$ coordinates

$$H_\phi = \frac{\sqrt{x^2+y^2}\ J}{2}$$

$$H_x = -H_\phi \sin\phi = \frac{\sqrt{x^2+y^2}\ J}{2} \frac{y}{\sqrt{x^2+y^2}} = -\frac{Jy}{2}$$

$$H_y = H_\phi \cos\phi = \frac{\sqrt{x^2+y^2}\ J}{2} \frac{x}{\sqrt{x^2+y^2}} = \frac{Jx}{2}$$

For a conductor centred at $x = d$ carrying current $-J$

$$H_x' = \frac{Jy}{2} \qquad H_y' = -\frac{J(x-d)}{2}$$

Total H is $H_x + H_x' = 0$ in x direction

$H_y + H_y'$ in y direction $= Jd$ $\quad \underline{B} = \mu_0 Jd\ \hat{y}$

(2) Cheng 6-17

Inside conductor $\underline{B} = B_\phi \hat{\underline{\phi}} = \frac{\mu_0 r I}{2\pi b^2} \hat{\underline{\phi}}$

Outside conductor $\underline{B} = B'_\phi \hat{\underline{\phi}} = \frac{\mu_0 I}{2\pi r} \hat{\underline{\phi}}$

$\underline{B} = \text{curl } \underline{A}$ only need ϕ component of $\underline{A}$

$$(\text{curl } \underline{A})_\phi = \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\underline{\phi}}$$

There is no variation with z, this is an infinitely long conductor

Therefore: Inside $-\frac{\partial A_z}{\partial r} = \frac{\mu_0 r I}{2\pi b^2}$

$$A_z = -\frac{\mu_0 r^2 I}{4\pi b^2} + C_1$$

Outside $-\frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$

$$A_z = -\frac{\mu_0 I}{2\pi} \ln r + C_2$$

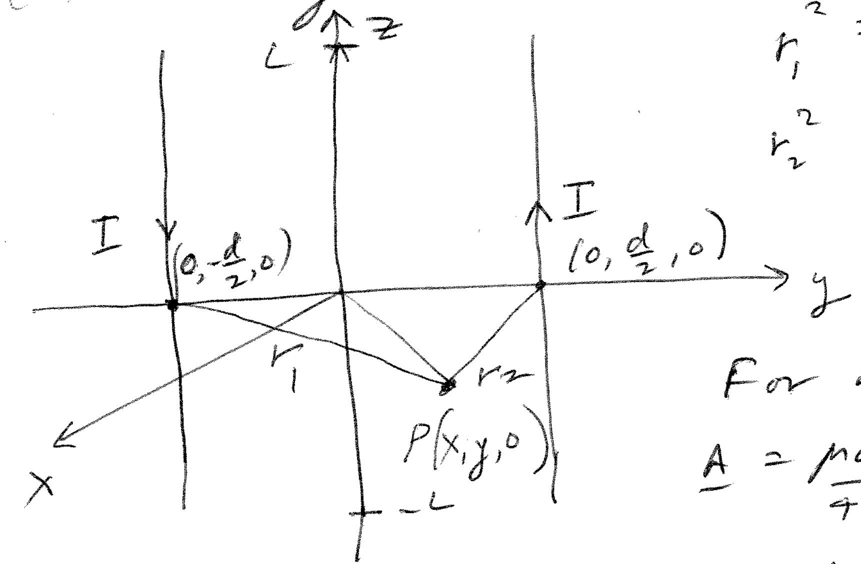
At edge of conductor $A_z(\text{inside}) = A_z(\text{outside})$

so $-\frac{\mu_0 I}{4\pi} + C_1 = -\frac{\mu_0 I}{2\pi} \ln b + C_2$

we need to choose a reference value, set $A_z = 0$ on surface of conductor. This gives

$$C_1 = \frac{\mu_0 I}{4\pi} \qquad C_2 = \frac{\mu_0 I}{2\pi} \ln b$$

(3) Cheng 6-18



$$r_1^2 = x^2 + \left(y+\frac{d}{2}\right)^2$$

$$r_2^2 = x^2 + \left(y-\frac{d}{2}\right)^2$$

For a single wire

$$\underline{A} = \frac{\mu_0 I}{4\pi} \ln\left(\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}\right)\hat{\underline{z}}$$

(a) For the 2 wires with opposite currents

$$\underline{A} = \frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{\sqrt{L^2+r_2^2}+L}{\sqrt{L^2+r_2^2}-L}\right) - \ln\left(\frac{\sqrt{L^2+r_1^2}+L}{\sqrt{L^2+r_1^2}-L}\right)\right]\hat{\underline{z}}$$

$$\underline{A} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{\sqrt{L^2+r_2^2}+L \cdot \sqrt{L^2+r_1^2}-L}{\sqrt{L^2+r_2^2}-L \cdot \sqrt{L^2+r_1^2}+L}\right]\hat{\underline{z}}$$

Write $\sqrt{L^2+r_2^2} = \sqrt{C}$ $\quad \sqrt{L^2+r_1^2} = \sqrt{D}$

$$\underline{A} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{(\sqrt{C}+L)(\sqrt{D}-L)}{(\sqrt{C}-L)(\sqrt{D}+L)}\right]\hat{\underline{z}}$$

$$= \frac{\mu_0 I}{4\pi}\ln\left[\frac{C-L^2}{(\sqrt{C}-L)^2}\,\frac{(\sqrt{D}-L)^2}{D-L^2}\right] = \frac{\mu_0 I}{4\pi}\ln\left[\frac{r_2^2(\sqrt{D}-L)^2}{r_1^2(\sqrt{C}-L)^2}\right]\hat{\underline{z}}$$

$$\underline{A} = \frac{\mu_0 I}{2\pi}\ln\left[\frac{r_2}{r_1}\,\frac{\sqrt{L^2+r_1^2}-L}{\sqrt{L^2+r_2^2}-L}\right]\hat{\underline{z}}$$

(b) As $L \to \infty$

$$\underline{A} = \frac{\mu_0 I}{2\pi} \ln\left[\frac{r_2}{r_1} \frac{\left\{L\left(1+\frac{r_1^2}{2L^2}\right) - L\right\}}{\left\{L\left(1+\frac{r_2^2}{2L^2}\right) - L\right\}} \right] = \frac{\mu_0 I}{2\pi} \ln\left(\frac{r_1}{r_2}\right) \hat{z}$$

$$\underline{A} = \frac{\mu_0 I}{2\pi} \ln\left[\frac{\sqrt{x^2 + (y+\frac{d}{2})^2}}{\sqrt{x^2 + (y-\frac{d}{2})^2}}\right] = \frac{\mu_0 I}{4\pi} \ln\left(\frac{r_1^2}{r_2^2}\right)$$

(c) $$\underline{B} = \operatorname{curl} \underline{A} = \frac{\partial A_z}{\partial y}\hat{\underline{i}} - \frac{\partial A_z}{\partial x}\hat{\underline{j}}$$

$$= \frac{\mu_0 I}{2\pi}\left[\frac{(y+\frac{d}{2})}{x^2 + (y+\frac{d}{2})^2} + \frac{(y-\frac{d}{2})}{(y-\frac{d}{2})^2 + x^2}\right]\hat{\imath}$$

$$- \frac{\mu_0 I}{2\pi}\left[\frac{x}{x^2 + (y+\frac{d}{2})^2} - \frac{x}{x^2 + (y-\frac{d}{2})^2}\right]\hat{\underline{j}}$$

gives $$\underline{B} = \frac{\mu_0 I}{2\pi}\left[\frac{\hat{\underline{\phi}}_1}{r_1} - \frac{\hat{\phi}_2}{r_2}\right]$$

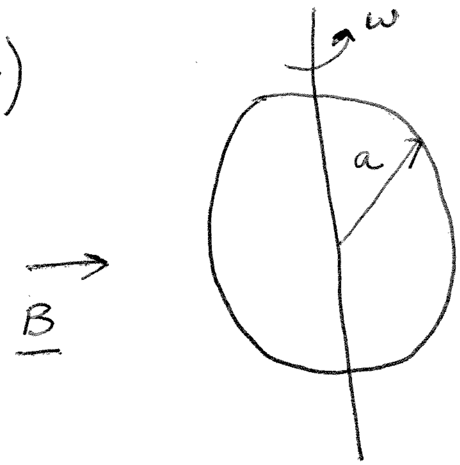
(d) For the flux lines in the xy plane

$$\frac{dy}{dx} = \frac{B_y}{B_x} = \frac{-\partial A_z/\partial x}{\frac{\partial A_z}{\partial y}}$$

gives $$\frac{\partial A_z}{\partial y} dy + \frac{\partial A_z}{\partial x} dx = 0$$ Therefore $\underline{A}$ = constant

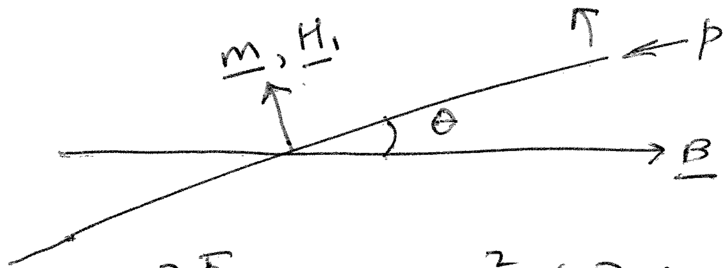
Therefore $$\frac{r_1^2}{r_2^2} = \text{constant} = \frac{x^2 + (y+\frac{d}{2})^2}{x^2 + (y-\frac{d}{2})^2}$$

(4)



The total flux through the coil as it rotates is

$$\Phi = \pi a^2 N B \sin\theta$$



$$\theta = \omega t$$

$$\frac{\partial\Phi}{\partial t} = -\pi a^2 N B \cos\theta \frac{d\theta}{dt} = -\pi a^2 N B \omega \cos\omega t$$

Now $\frac{\partial\Phi}{\partial t} = -\oint E \cdot dl = emf = V$

The current induced in the coil is

$$I = \frac{V}{R} = \frac{\pi a^2 N B \omega \cos\omega t}{R}$$

The ohmic heating is $VI = \frac{N^2\pi^2 a^4 B^2 \omega^2 \cos^2\omega t}{R}$

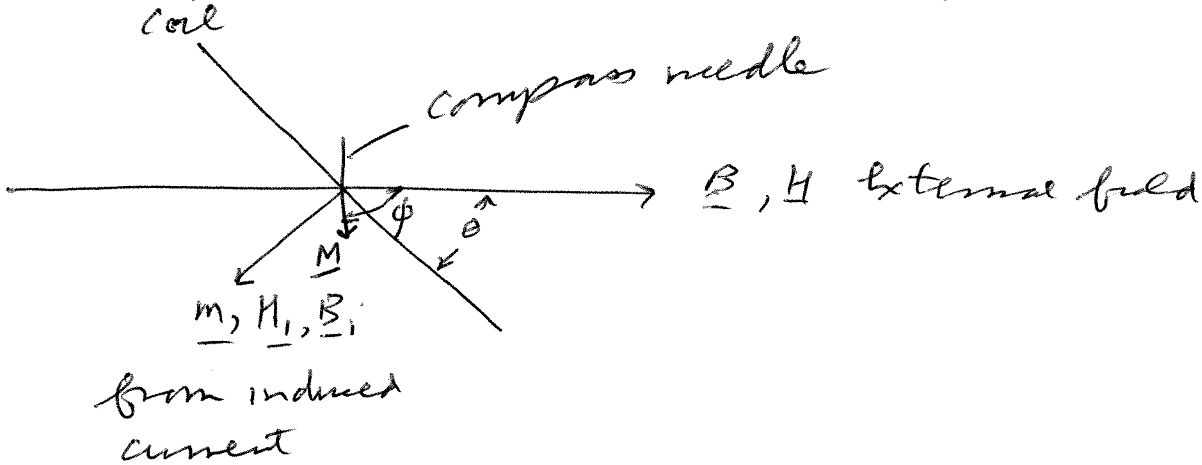
The average ohmic heating is $\frac{N^2\pi^2 a^4 B^2 \omega^2}{2R}$

The current in the coil produces a magnetic dipole $\underline{m}$ that is oriented as shown. According to Lenz's law it acts to oppose the change in external flux

The magnetic dipole of the induced current is

$$\underline{m} = I N \underline{A} = \pi a^2 N I$$

$\uparrow$ area of coil (vector)



Two magnetic fields act on the small magnetic compass needle – the external field, and the field from the induced current

The field from the induced current is

$$B_1 = \frac{\mu_0 N I}{2a} \quad \text{(at the center of the coil)}$$

$$B_1 = \frac{\mu_0 \pi a N^2 \omega B \cos\omega t}{2R} = k B \cos\omega t$$

where $k = \dfrac{\mu_0 \pi a N^2 \omega}{2R}$

Two torques act on the needle. In equilibrium

$$\underline{M} \times \underline{B}_1 = \underline{M} \times \underline{B}$$

This gives

$$\underline{M}\ \underline{B}_1 \sin(90-(\phi-\omega t)) = MB\sin\phi$$

Therefore

$$k\cos(\omega t)(\cos\phi\cos\omega t + \sin\phi\sin\omega t) = \sin\phi$$

$$k\left[\cos\phi\cos^2(\omega t)\right] + k\left[\sin\phi\sin\theta\cos\theta\right] = \sin\phi$$

gives $\frac{k\cos\phi}{2} = \sin\phi$

$$\tan\phi = \frac{k}{2}$$

with $a = 100$ mm, $N = 100$, $\omega = 100$

$B = 0.01$T $\quad R = 5$ ohm

The energy dissipated in the coil is

$\underline{0.987\ W}$

$$\phi = \underline{1.131^\circ}$$

(5)

$v := 10^{6}$ electron velocity

$e := 1.6 \cdot 10^{-19}$ magnitude of electronic charge

$\varepsilon_0 := 8.854 \cdot 10^{-12}$ permittivity of free space

$i := 1, 2 .. 100$

$t_i := -10 \cdot 10^{-9} + \frac{(i-1) \cdot 20 \cdot 10^{-9}}{99}$ various times

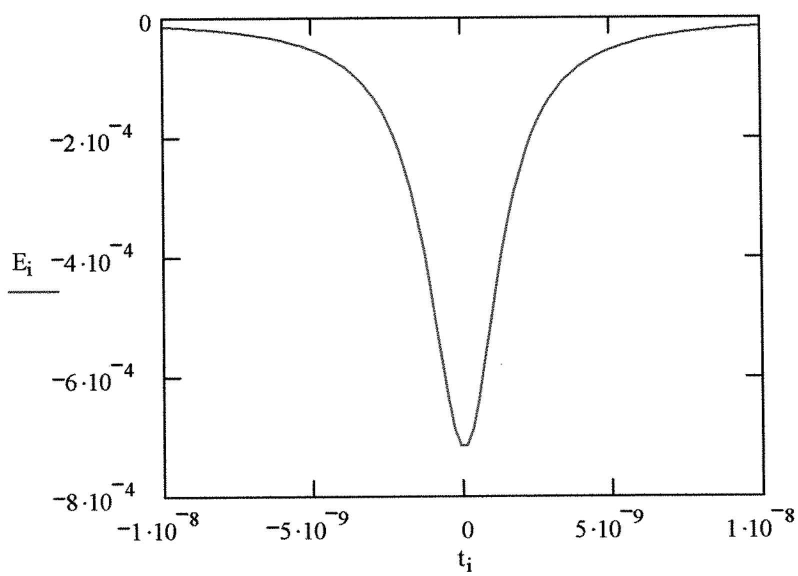
$z_i := t_i \cdot v$ various positions of electrons

$x := 10^{-3}$ $y := 10^{-3}$ x,y coordinates in m

$r_i := \sqrt{x^2 + y^2 + \left(z_i\right)^2}$ distances from origin

$$E_i := \frac{-e}{4 \cdot \pi \cdot \varepsilon_0 \cdot \left(r_i\right)^2}$$

Magnitude of electric field v. time



Now consider a line of electrons

$j := 1, 2 .. 100$ running integer for 100 electrons

$z_{i,j} := t_i \cdot v - \left[(j - 50) \cdot 10^{-3} + 10^{-6} \right]$ various positions of electrons

$x := 10^{-3}$ $y := 10^{-3}$ x,y coordinates

$r_{i,j} := \sqrt{x^2 + y^2 + \left(z_{i,j}\right)^2}$ distances from origin

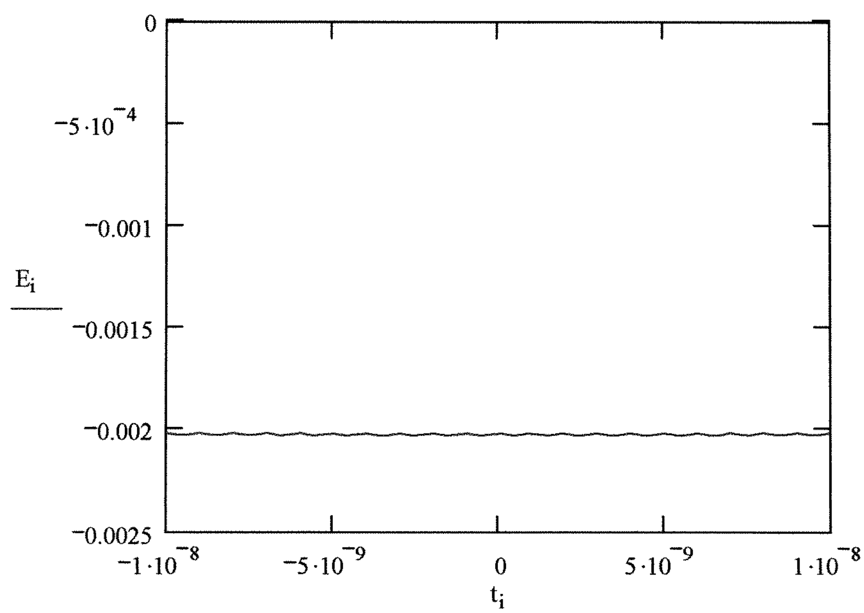
$\theta_{i,j} := \operatorname{atan}\left(\frac{\sqrt{2} \cdot 10^{-3}}{z_{i,j}} \right)$ angles to each electron

$Ez_i := \sum_{j=1}^{100} \frac{-e}{4 \cdot \pi \cdot \varepsilon_0 \cdot \left(r_{i,j}\right)^2} \cdot \cos\left(\theta_{i,j}\right)$ electric field in z direction

$Exy_i := \sum_{j=1}^{100} \frac{-e}{4 \cdot \pi \cdot \varepsilon_0 \cdot \left(r_{i,j}\right)^2} \cdot \sin\left(\theta_{i,j}\right)$ electric field in xy plane

$E_i := -\sqrt{\left(Ez_i\right)^2 + \left(Exy_i\right)^2}$

Magnitude of electric field v. time



Note small ripple in electric field because of the discrete electrons

The equivalent current is found from the charge passing the z=0 plane per second

The number of electrons crossing the z=0 plane per second is

$$N := \frac{v}{10^{-3}}$$

$$N = 1 \times 10^{9}$$

$I := N \cdot e$ current

$R := \sqrt{2 \cdot 10^{-3}}$ distance of line of charge from (0,0,0)

Magnetic field magnitude H is

$$H_{\phi} := \frac{I}{2 \cdot \pi \cdot R}$$ found from Amperes law

$H_{\phi} = 5.694 \times 10^{-10}$ A/m -- points in the azimuthal direction

(6) The Lorentz force on the electrons in the metal disk is
F=-e(**v X B**)

This force is equivalent to a "motional electric field" of
$\mathbf{E_m}$=**v X B**

$B := 0.1$ magnetic flux density

$R := \frac{0.5}{2}$ radius of disk

$\omega := 1000 \cdot 60 \cdot 2 \cdot \pi$ angular velocity of disk

$v = \omega \cdot r$ velocity at radius r

$E_m(r) = \omega \cdot r \cdot B$ motional electric field at radius r, acts radially

$V := -\int_0^R \omega \cdot r \cdot B \, dr$ Potential difference between center of disk and edge of disk

$V = -1.178 \times 10^{3}$

(7) **Integrate over current distribution to find A**

$$A = \int \frac{\mu_0 \cdot J}{4 \cdot \pi \cdot r} \, dV$$

For a point P at (R,ϕ,z), the vector potential clearly does not depend on z or ϕ

$dV = 2 \cdot \pi \cdot r \cdot dr \cdot dz$ volume element

For a point at radius R, (Cartesian cordinates (R,0,0)) and a cylindrical volume element at (r1,ϕ,z) the separation distance is

$$r = \sqrt{R^2 - 2 \cdot r1 \cdot R \cdot \cos(\phi) + r1^2 + z^2}$$

$\text{radius} := 5 \cdot 10^{-3}$ $\mu_0 := 4 \cdot \pi \cdot 10^{-7}$

Integral is

$$A_z = \int_0^{\text{radius}} \int_{-\infty}^{\infty} \int_0^{2 \cdot \pi} \frac{\mu_0 \cdot 10 \cdot e^{-10^5 \cdot r1^2} \cdot 2 \cdot \pi \cdot r1}{4 \cdot \pi \cdot \sqrt{R^2 - 2 \cdot r1 \cdot R \cdot \cos(\phi) + r1^2 + z^2}} \, d\phi \, dz \, dr1$$

A_z clearly only varies with R

Inside or outside the conductor

$$A_z = \int_0^{\text{radius}} \int_{-\infty}^{\infty} \int_0^{2 \cdot \pi} \frac{\mu_0 \cdot 10 \cdot e^{-10^5 \cdot r1^2} \cdot 2 \cdot \pi \cdot r1}{4 \cdot \pi \cdot \sqrt{R^2 - 2 \cdot r1 \cdot R \cdot \cos(\phi) + r1^2 + z^2}} \, d\phi \, dz \, dr1$$

B=curl(**A**) $\mathrm{Curl}(A_z) = \frac{d}{dR} A_z$ points in the phi direction

$$B_\phi = \frac{d}{dR} A_z$$

Inside or outside the conductor

$R := 5\cdot 10^{-3}$

$$A_z(R) := \int_0^{radius}\int_{-\infty}^{\infty}\int_0^{2\cdot\pi} \frac{\mu_0\cdot 10\cdot e^{-10^5\cdot r1^2}\cdot 2\cdot\pi\cdot r1}{4\cdot\pi\cdot\sqrt{R^2 - 2\cdot r1\cdot R\cdot\cos(\phi) + r1^2 + z^2}}\, d\phi\, dz\, dr1$$

$$B_\phi(R) := \frac{d}{dR}\left(\int_0^{radius}\int_{-\infty}^{\infty}\int_0^{2\cdot\pi} \frac{\mu_0\cdot 10\cdot e^{-10^5\cdot r1^2}\cdot 2\cdot\pi\cdot r1}{4\cdot\pi\cdot\sqrt{R^2 - 2\cdot r1\cdot R\cdot\cos(\phi) + r1^2 + z^2}}\, d\phi\, dz\, dr1\right)$$

These integrals are not simple to evaluatre

Alternative solution found by integrating over current enclosed $B=\mu_0 I_{enclosed}/(2\pi R)$

Outside the conductor

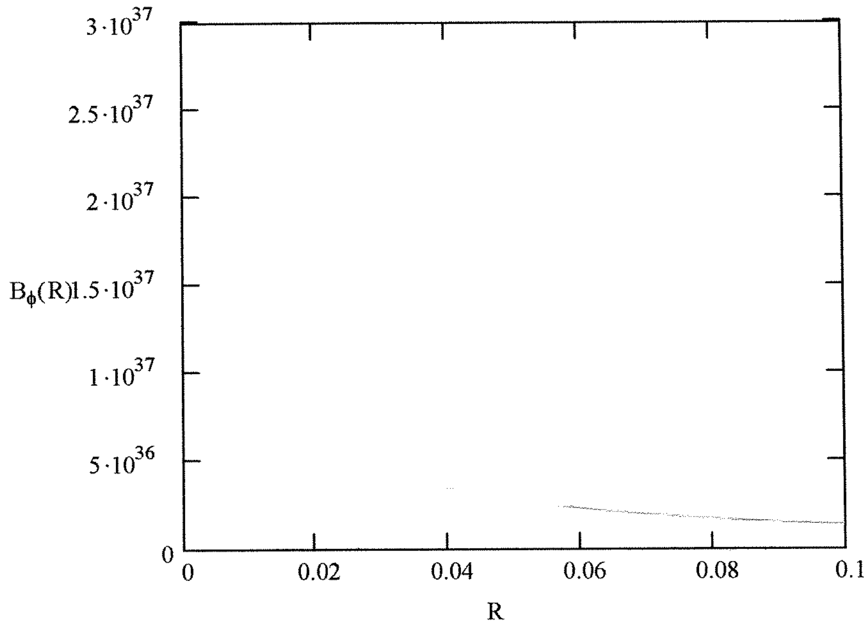
$R := 5\cdot 10^{-3}, 5.1\cdot 10^{-3} .. 100\cdot 10^{-3}$ various values for R

$$I_{enclosed} := \int_0^{radius} \mu_0\cdot 10\cdot e^{-10^5\cdot r^2}\cdot 2\cdot\pi\cdot r\, dr$$

$I_{enclosed} = 8.908\times 10^{35}$ A very large current with the values given

$$B_\phi(R) := \frac{\int_0^{radius} \mu_0\cdot 10\cdot e^{-10^5\cdot r^2}\cdot 2\cdot\pi\cdot r\, dr}{2\cdot\pi\cdot R}$$

Outside

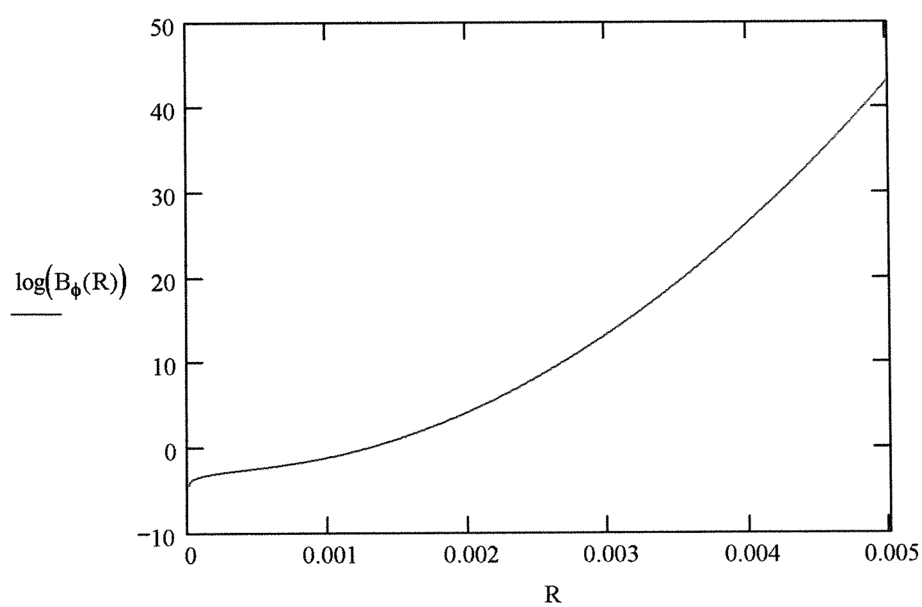


Inside the conductor

$R := 0.01 \cdot 10^{-3}, 0.02 \cdot 10^{-3} .. 5 \cdot 10^{-3}$ various values of R

$$B_{\phi}(R) := \frac{\int_{0}^{R} 10 \cdot e^{-10^5 \cdot r^2} \cdot 2 \cdot \pi \cdot r \, dr}{2 \cdot \pi \cdot R}$$

Inside



(8) $\underline{v} = v_0 \hat{\underline{k}}$ initially

$\underline{B} = B_0 \hat{\underline{j}}$ $\quad \underline{E} = E_0 \hat{\underline{i}}$

Velocity is $\underline{v} = v_x \hat{\underline{i}} + v_y \hat{\underline{j}} + v_z \hat{\underline{k}}$

Lorentz force is $\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$

$$\underline{F} = q(E_0 \hat{\underline{i}} + (v_x \hat{\underline{i}} + v_y \hat{\underline{j}} + v_z \hat{\underline{k}}) \times B_0 \hat{\underline{j}})$$

$$= q(E_0 - v_z B_0) + q v_x B_0$$

If the particle has mass m

$$\frac{dv_x}{dt} = \frac{q}{m}(E_0 - v_z B_0) \qquad \frac{dv_z}{dt} = \frac{q}{m} v_x B_0 = \omega v_x$$

where $\omega = \frac{q B_0}{m}$

$$\frac{d^2 v_x}{dt^2} = -\frac{q B_0}{m}\frac{dv_z}{dt} = -\omega^2 v_x$$

$$v_x = A \sin(\omega t) + B \cos(\omega t)$$

At time $t=0$, $v_x = 0$, so $B = 0$

$$v_x = A \sin \omega t \qquad \frac{dv_x}{dt} = A\omega \cos \omega t$$

At time $t=0$ $\frac{dv_x}{dt} = \frac{q}{m}(E_0 - v_0 B_0) = \omega\left(\frac{E_0}{B_0} - v_0\right)$

so $A = \left(\frac{E_0}{B_0} - v_0\right)$

$$v_x = \left(\frac{E_0}{B_0} - v_0\right) \sin \omega t$$

$$x = -\frac{1}{\omega}\left(\frac{E_0}{B_0} - v_0\right) \cos \omega t + C$$

at $t=0$ $x=0$, so $C = \frac{1}{\omega}\left(\frac{E_0}{B_0} - V_0\right)$

$$x = \frac{1}{\omega}\left(\frac{E_0}{B_0} - V_0\right)(1 - \cos\omega t)$$

$$\frac{dV_z}{dt} = \omega\left(\frac{E_0}{B_0} - V_0\right)\sin\omega t$$

$$V_z = -\left(\frac{E_0}{B_0} - V_0\right)\cos\omega t + D$$

At $t=0$ $V_z = V_0$ so $D = \left(\frac{E_0}{B_0}\right)$

$$V_z = \frac{E_0}{B_0} - \left(\frac{E_0}{B_0} - V_0\right)\cos\omega t$$

$$z = \frac{E_0 t}{B_0} - \frac{1}{\omega}\left(\frac{E_0}{B_0} - V_0\right)\sin\omega t + E$$

At $t=0$ $z=0$, so $E=0$

$$z = \frac{E_0 t}{B_0} - \frac{1}{\omega}\left(\frac{E_0}{B_0} - V_0\right)\sin\omega t$$

See examples of what trajectories look like on following pages

(8) Examples of particle trajectories

Choose some values

$q := -1.6\cdot 10^{-19}$

electron

$m := 9.11\cdot 10^{-31}$

$E_0 := 10$

$B_0 := 10^{-3}$

$v_0 := 0$ An example where the electron is at rest at t=0

$i := 1, 2 .. 1000$

various times on a nanosecond scale

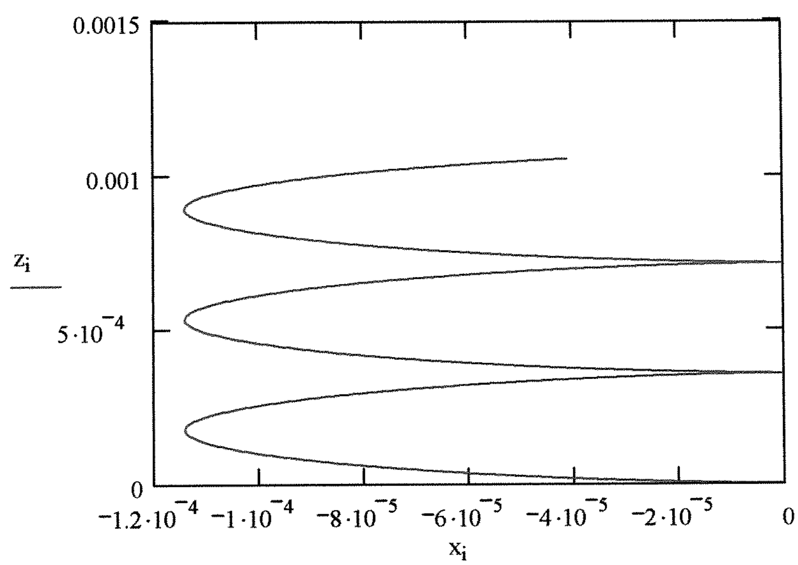
$t_i := i\cdot 10^{-10}$

$\omega := q\cdot\frac{B_0}{m}$

$\omega = -1.756\times 10^{8}$

$$x_i := \frac{1}{\omega}\cdot\left(\frac{E_0}{B_0} - v_0\right)\cdot\left(1 - \cos\left(\omega\cdot t_i\right)\right)$$

$$z_i := \frac{E_0}{B_0}\cdot t_i - \frac{1}{\omega}\cdot\left(\frac{E_0}{B_0} - v_0\right)\cdot\sin\left(\omega\cdot t_i\right)$$



This is the trajectory when the particle stars from rest -- it is callled a cycloid

$E_0 := 0$ **No electric field**

$B_0 := 10^{-3}$

$v_0 := 100$

$i := 1, 2 .. 1000$

various times on a nanosecond scale

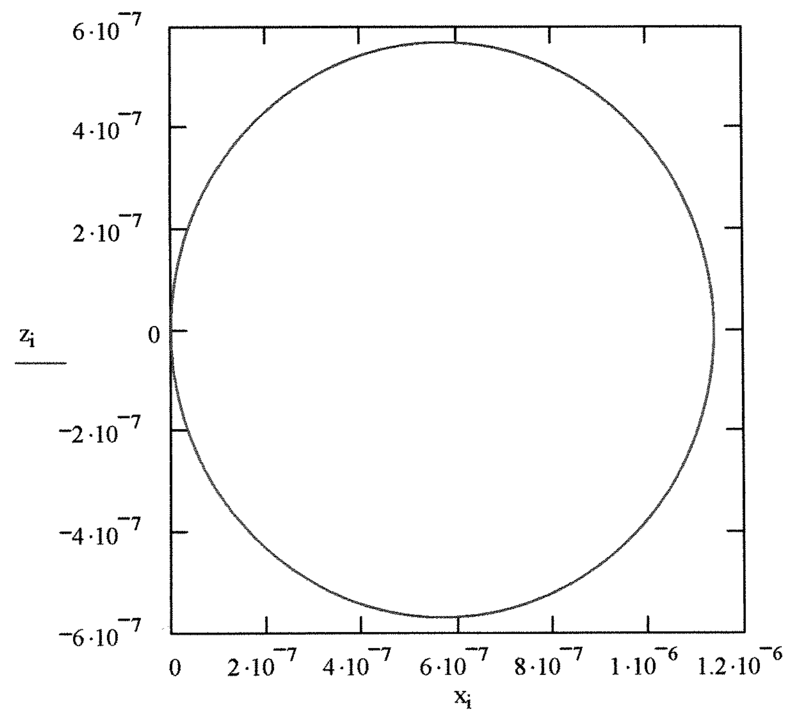
$t_i := i \cdot 10^{-10}$

$$\omega := q \cdot \frac{B_0}{m}$$

$$\omega = -1.756 \times 10^{8}$$

$$x_i := \frac{1}{\omega} \cdot \left(\frac{E_0}{B_0} - v_0 \right) \cdot \left(1 - \cos\left(\omega \cdot t_i\right) \right)$$

$$z_i := \frac{E_0}{B_0} \cdot t_i - \frac{1}{\omega} \cdot \left(\frac{E_0}{B_0} - v_0 \right) \cdot \sin\left(\omega \cdot t_i\right)$$



Particle moves in a circle