ENEE 381 Final Examination
December 17, 2004, 1:30- 3:30 pm
ANSWER 4 QUESTIONS
If more than 4 are answered, best 4 will count

(1) Explain the experimental observations that underlie Maxwell's equations, especially explaining why the displacement current was needed to modify Ampères Law. (3pts.) Starting from Maxwell's equations show that the electric and magnetic fields in a uniform medium satisfy the 3-D wave equation. (2pts.)

A plane wave has magnetic flux density $B_x = 100\mu_0 \cos(3\pi 10^7 t - kz)$ and is traveling in a medium with ϵ_r=9, μ_r=4. What are the electric field amplitude, wavelength, propagation constant, impedance and magnetic field amplitude of the wave? (3pts.) If the plane wave in this medium had $B_x = 100\mu_0 \cos(3\pi 10^7 t - kz)$, and the electric field were 45° out of phase, what would be the average Poynting vector? (1pt.) In what situations is the Poynting vector zero? (1pt.)

(2) Prove that the reflection coefficient for a current wave on a transmission line of characteristic admittance Y_0 when it strikes a load with admittance Y_L is (3pts.)

$$\rho_I = \frac{Y_L - Y_0}{Y_L + Y_0}.$$

How does the reflection coefficient for the current differ from the reflection coefficient for the voltage?(1pt.)

A 50ohm transmission line is terminated with a load of 75-j75 ohm. Where on the line nearest to the load can you match the line with a parallel inductor. If the operating frequency of the line is 1 GHz and the phase velocity on the line is 2×10^8 m/s what is the value of the inductor and how far from the load is it placed? You may use the Smith Chart to solve this problem. (6pts.)

(3) A TE wave is incident on a dielectric boundary between 2 media of dielectric constant ϵ_{r1} and ϵ_{r2} at an angle θ_1. Prove that the angle of reflection equals the angle of incidence and that the angle of refraction obeys Snell's Law. (2pts.) By matching field components parallel to the boundary show that the reflection coefficient is (3pts.)

$$\rho = \frac{\frac{Z_2}{n_2 \cos(\theta_2)} - \frac{Z_1}{n_1 \cos(\theta_1)}}{\frac{Z_2}{n_2 \cos(\theta_2)} + \frac{Z_1}{n_1 \cos(\theta_1)}}.$$

A TE wave of intensity 1W/m^2 strikes a boundary between 2 lossless dielectrics with $\epsilon_{r1} = 4$, ϵ_{r2}=2, at an angle of 30°. What is the intensity of the transmitted wave? (3pts.) Where is the nearest magnetic field maximum to the boundary that is not at the boundary? (1pt.) What is the standing wave ratio in medium 1 (1pt.)?

(4) A TM_1 wave is propagating between perfectly conducting parallel infinite planes separated by distance a and filled with a dielectric ϵ_r. Derive all the field components of the wave (3pts.) and prove that the characteristic impedance of the mode is (3pts.)

$$Z_{TM} = Z_{medium}\sqrt{1-\left(\frac{\omega_c}{\omega}\right)^2}.$$

If the frequency of the wave is twice the cutoff frequency, a=10mm, ϵ_r=2, what is the effective angle of the zig-zag TEM wave that is an alternative description of the wave? (4pts.)

Hint:

$$E_x = -\frac{\gamma}{K^2}\frac{\partial E_z}{\partial x}.$$

$$H_y = -\frac{j\omega\epsilon\epsilon_r}{K^2}\frac{\partial E_z}{\partial x}.$$

(5) Prove that the cutoff frequency of a TE mode in a rectangular waveguide of dimension $a \times b$ is (4pts.)

$$\nu_c = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

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Hint:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = -(\gamma^2 + k^2)E_z.$$

For ν=10GHz, a=50mm, b=20mm, and a TE_{11} mode what are the values of β, λ_g, v_p, and v_g? (4pts.)

Sketch the $\omega - \beta$ behavior above ω_c and $\omega - \alpha$ below ω_c. (2pts.)

(6) A TE_{12} mode is propagating at frequency 20GHz in a waveguide with a=100mm, b=50mm. The wave enters a second part of the waveguide that is filled with a lossless dielectric of ϵ_r=2. What are $|\rho|$, ϕ and the VSWR? (8pts.)Where is the nearest electric field maximum to the boundary? (2pts.)

Hint: This is analagous to a transmission line problem. Information given in question (5) should be helpful. Also:

$$Z_{TE} = \frac{Z_{medium}}{\sqrt{1-(\omega_c/\omega)^2}}.$$

$$\epsilon_0 = 8.854 \times 10^{-12} \quad \text{F/m}.$$

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{Henry/m}.$$

$$c_0 = 2.998 \times 10^8 \quad \text{m/s}.$$

$$Z_0 = \sqrt{\mu_0/\epsilon_0} = 376.7 \text{ ohm}.$$

$$\epsilon'' = \sigma/\omega\epsilon_0.$$

Maxwell's equations

$$\mathrm{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

$$\mathrm{curl}\mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}.$$

$$\mathrm{div}\mathbf{D} = \rho.$$

$$\mathrm{div}\mathbf{B} = 0.$$

$$k = \omega\sqrt{\epsilon_0\epsilon_r\mu_0\mu_r}.$$

$$k = \omega/c.$$

$$\mathrm{curl\ curl}\mathbf{E} = \mathrm{grad\ div}\mathbf{E} - \nabla^2\mathbf{E}.$$

$$Z_0 = \sqrt{\mu_0/\epsilon_0} = 376.7\ \mathrm{ohm}.$$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}.$$

$$\mathbf{CurlE} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{\mathbf{k}}.$$

ENEE 381 Fall 2004 FINAL EXAM SOLUTIONS

(1) Experimental observations and equations

Gauss's Law for electric charges $\mathrm{div}(D) = \rho$

Gauss's Law for magnetics $\mathrm{div}(B) = 0$

Faraday's Law plus Lenz's Law $\mathrm{curl}(E) = -\left(\frac{\partial}{\partial t}B\right)$

Amperes Law plus Displacement Current $\mathrm{curl}(H) = j + \frac{\partial}{\partial t}D$

Use vector identities, with j=0

$$\mathrm{curlcurl}(E) = \mathrm{graddiv}(E) - \mathrm{del}^2(E) = \frac{\partial}{\partial t}\mathrm{curl}(B) = -\mu_0\cdot\mu_r\cdot\mathrm{curl}(H) = -\mu_0\cdot\mu_r\cdot\varepsilon_0\cdot\varepsilon_r\cdot\frac{d^2}{dt^2}E$$

When ρ=0 div(E)=0 and this reduces to the wave equation

$\omega := 3\cdot\pi\cdot 10^7$ angular frequency

$\varepsilon_0 := 8.854\cdot 10^{-12}$ $\mu_0 := 4\cdot\pi\cdot 10^{-7}$

$\varepsilon_r := 9$ $\mu_r := 4$ $B := 100\cdot\mu_0$

$B = 1.257\times 10^{-4}$ Tesla

$Z := \sqrt{\frac{\mu_0\cdot\mu_r}{\varepsilon_0\cdot\varepsilon_r}}$ $Z = 251.156$ impedance, ohms

$H := 100\cdot\frac{\mu_0}{\mu_0\cdot\mu_r}$ $H = 25$ magnetic field amplitude A/m

$E := H\cdot Z$

$E = 6.279\times 10^3$ Electric field amplitude V/m

$\nu := \frac{\omega}{2\cdot\pi}$ $c := \frac{2.998\cdot 10^8}{\sqrt{\varepsilon_r\cdot\mu_r}}$ $c = 4.997\times 10^7$ Velocity of light in medium

$\lambda := \frac{c}{\nu}$ $\lambda = 3.331$ Wavelength (m)

$k := 2\cdot\frac{\pi}{\lambda}$ $k = 1.886$ Propagation constant m^{-1}

If **E** and **H** are 45 degrees out of phase

$E_1 := E \cdot e^{i \cdot 45 \cdot deg}$

$S_{avg} := 0.5 \cdot Re\left(E_1 \cdot \overline{H}\right)$

$S_{avg} = 5.55 \times 10^4$ Average Poynting vector W/m^2

Average Poynting vector is zero if **E** and **H** are parallel or if **E** and **H** are 90 degrees out of phase

(2) At the load

$$V_L = V_i + V_r = \frac{I_i}{Y_0} - \frac{I_r}{Y_0}$$

$$I_L = I_i + I_r$$

$$\frac{I_L}{V_L} = Y_L$$ which gives

$$\frac{(I_i + I_r) \cdot Y_0}{I_i - I_r} = Y_L$$ which gives

$$(I_i + \rho_I \cdot I_i) \cdot Y_0 = (I_i - \rho_I \cdot I_i) \cdot Y_L$$ which gives

$$\rho_I = \frac{Y_L - Y_0}{Y_L + Y_0}$$

This is 180 degrees different from the reflection coefficient for the voltage

Transmission line problem

$Z_L := 75 - i \cdot 75$

$Z_0 := 50$ $Y_0 := \frac{1}{Z_0}$

$\zeta_L := \frac{Z_L}{Z_0}$ $\zeta_L = 1.5 - 1.5i$ Normalized load. Pt. A on Smith Chart

$y_L := \frac{1}{\zeta_L}$ $y_L = 0.333 + 0.333i$ Normalized admittance of load. Pt B

For an inductor $Y_L = \frac{-j}{\omega \cdot L}$ so match at Pt C, 1+j1.27

Point C is 0.171λ-0.055λ=0.116λ from normalized admitttance of load

$\omega := 2 \cdot \pi \cdot 10^9$ $v_p := 2 \cdot 10^8$

$$\lambda := \frac{v_p}{10^9} \qquad \lambda = 0.2$$

Distance to match point is

$$d := 0.116 \cdot \lambda \qquad d = 0.023 \qquad \text{23mm}$$

$$\frac{\frac{1}{\omega \cdot L}}{Y_0} = 1.27 \qquad L := \frac{1}{1.27 \cdot \omega \cdot Y_0}$$

$$L = 6.266 \times 10^{-9} \qquad \text{6.266 nanoHenry}$$

(3) Phase factor variation parallel to boundary must be the same for incident, reflected and transmitted waves

$$k_1 \cdot \sin(\theta_{inc}) = k_1 \cdot \sin(\theta_{refl}) = k_2 \cdot \sin(\theta_{trans})$$

This proves that $\theta_{inc}=\theta_{refl}$, and Snell's law

$$Z_0 := 376.7$$

$$\theta_1 := 30 \cdot deg \qquad n_1 := \sqrt{4} \qquad n_2 := \sqrt{2}$$

$$\theta_2 := \operatorname{asin}\left(\frac{n_1 \cdot \sin(\theta_1)}{n_2}\right)$$

$$\frac{\theta_2}{deg} = 45$$ angle of refraction

For a TE wave $\quad Z_1 := \frac{Z_0}{n_1 \cdot \cos(\theta_1)} \qquad Z_2 := \frac{Z_0}{n_2 \cdot \cos(\theta_2)}$

$$\rho := \frac{Z_2 - Z_1}{Z_2 + Z_1} \qquad \rho = 0.268 \qquad T := 1 - (|\rho|)^2$$

$T = 0.928$ Transmitted wave intensity is 0.928 W/m^2

Boundary is a electric field minimum, a magnetic field maximum, so nearest magnetic field maximum that is not at the boundary is 1/4 effective wavelength away. Propagation constant normal to boundary is

$$k_{norm} = \frac{2 \cdot \pi}{\lambda} \cdot \cos(\theta_1)$$

Effective distance is $d = \frac{\lambda}{4 \cdot \cos(\theta_1)}$

$$\frac{1}{4 \cdot \cos(\theta_1)} = 0.289$$

0.289 λ away from boundary

(4) $E_z = A\cdot\sin\left(\frac{\pi\cdot x}{a}\right)$

$$\gamma = \sqrt{\left(\frac{\pi}{a}\right)^2 - \left(\frac{\omega}{c}\right)^2} = \frac{\omega}{c}\cdot\sqrt{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

$$\beta = \frac{\omega}{c}\cdot\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

$$E_x = \frac{-\gamma\cdot a}{\pi}\cdot A\cdot\cos\left(\frac{\pi x}{a}\right)$$

$$H_y = -j\cdot\frac{\omega\cdot\varepsilon_0\cdot\varepsilon_r\cdot a}{\pi}\cdot A\cdot\cos\left(\frac{\pi\cdot x}{a}\right)$$

$$\frac{E_x}{H_y} = \frac{\gamma}{j\cdot\omega\cdot\varepsilon_0\cdot\varepsilon_r} = \frac{\frac{\omega}{c}\cdot\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}}{\omega\cdot\varepsilon_0\cdot\varepsilon_r} = \sqrt{\frac{\mu_0\cdot\mu_r}{\varepsilon_0\cdot\varepsilon_r}}\cdot\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2} = Z_{medium}\cdot\sqrt{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

Cutoff frequency is $\omega_c = \frac{\pi\cdot c}{a}$

$c := \frac{2.998\cdot 10^8}{\sqrt{2}}$ $\quad a := 10^{-2}$

$\omega_c := \frac{\pi\cdot c}{a}$

$\nu_c := \frac{\omega_c}{2\cdot\pi}$ $\quad \nu_c = 1.06\times 10^{10}$

$\nu := 2\cdot\nu_c$

$\lambda := \frac{c}{\nu}$

The effective zig-zag angle satisfies

$$a = \frac{\lambda}{2\cdot\cos(\theta)}$$

$$\theta := \operatorname{acos}\left(\frac{\lambda}{2\cdot a}\right)$$

$$\frac{\theta}{deg} = 60$$

(5) In this case

$$K^2 = \left(\frac{m\cdot\pi}{a}\right)^2 + \left(\frac{n\cdot\pi}{b}\right)^2$$

$$\omega_c = c\cdot\sqrt{\left(\frac{m\cdot\pi}{a}\right)^2 + \left(\frac{n\cdot\pi}{b}\right)^2}$$

$$\nu_c = \frac{c}{2}\cdot\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \nu := 10^{10}$$

$$c := 2.998\cdot 10^8 \quad a := 50\cdot 10^{-3} \quad b := 20\cdot 10^{-3} \quad m := 1 \quad n := 1$$

$$\nu_c := \frac{c}{2}\cdot\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \nu_c = 8.072\times 10^9$$

$$\beta := \frac{2\cdot\pi\,\nu}{c}\cdot\sqrt{1-\left(\frac{\nu_c}{\nu}\right)^2} \qquad \beta = 123.7$$

$$\lambda_g := \frac{2\cdot\pi}{\beta} \qquad \lambda_g = 0.051$$

$$v_p := \frac{2\cdot\pi\cdot\nu}{\beta} \qquad v_p = 5.079\times 10^8$$

$$v_g := c\cdot\sqrt{1-\left(\frac{\nu_c}{\nu}\right)^2} \qquad v_g = 1.77\times 10^8$$

$$i := 1, 2 .. 1000$$

$$\nu_i := \frac{2\cdot\nu_c\cdot i}{1000}$$ different frequencies

$$\beta_i := \frac{2\cdot\pi\,\nu_i}{c}\cdot\sqrt{1-\left(\frac{\nu_c}{\nu_i}\right)^2}$$

(5) Solution for ν=20GHz

$\nu := 2\cdot 10^{10}$

$c := 2.998\cdot 10^{8}$ $a := 50\cdot 10^{-3}$ $b := 20\cdot 10^{-3}$ $m := 1$ $n := 1$

$\nu_c := \frac{c}{2}\cdot\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ $\lambda := \frac{c}{\nu}$ $\lambda = 0.015$

$\nu_c = 8.072\times 10^{9}$

$\beta := \frac{2\cdot\pi\,\nu}{c}\cdot\sqrt{1-\left(\frac{\nu_c}{\nu}\right)^2}$ $\beta = 383.5$

$\lambda_g := \frac{2\cdot\pi}{\beta}$ $\lambda_g = 0.016$

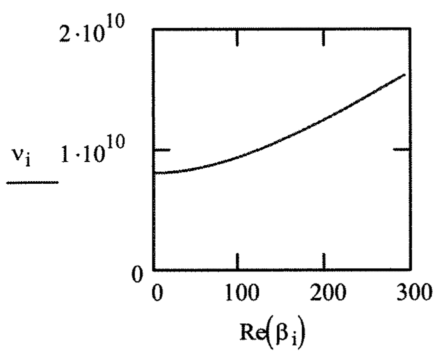
$v_p := \frac{2\cdot\pi\cdot\nu}{\beta}$ $v_p = 3.277\times 10^{8}$

$v_g := c\cdot\sqrt{1-\left(\frac{\nu_c}{\nu}\right)^2}$ $v_g = 2.743\times 10^{8}$

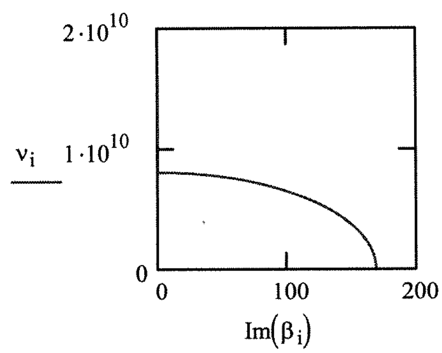
$i := 1, 2 .. 1000$

$\nu_i := \frac{2\cdot\nu_c\cdot i}{1000}$ different frequencies

$\beta_i := \frac{2\cdot\pi\,\nu_i}{c}\cdot\sqrt{1-\left(\frac{\nu_c}{\nu_i}\right)^2}$



Behavior of β above cutoff



Behavior of α below cutoff

(6) $a := 100\cdot10^{-3}$ $b := 50\cdot10^{-3}$ $c0 := 2.998\cdot10^{8}$

$m := 1$ $n := 2$ $\nu := 20\cdot10^{9}$

$$\nu_{c1} := \frac{c0}{2}\cdot\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$ cutoff frequency in air-filled waveguide

$\nu_{c1} = 6.181\times10^{9}$

$$c := \frac{c0}{\sqrt{2}}$$

$$\nu_{c2} := \frac{c}{2}\cdot\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$ $\nu_{c2} = 4.37\times10^{9}$

$$Z_1 := \frac{376.7}{\sqrt{1-\left(\frac{\nu_{c1}}{\nu}\right)^2}} \qquad Z_2 := \frac{\frac{376.7}{\sqrt{2}}}{\sqrt{1-\left(\frac{\nu_{c2}}{\nu}\right)^2}}$$

$Z_1 = 396.087$ $Z_2 = 272.964$

$$\rho := \frac{Z_2 - Z_1}{Z_2 + Z_1}$$ $\rho = -0.184$ $\phi := \arg(\rho)$

$\frac{\phi}{deg} = 180$

$|\rho| = 0.184$

$$S := \frac{1+|\rho|}{1-|\rho|}$$ $S = 1.451$

The boundary is an electric field minimum. Think about analogy with Smith chart. Nearest electric field maximum is $\lambda_g/4$ away from boundary

$$\beta := \frac{2\cdot\pi\,\nu}{c0}\cdot\sqrt{1-\left(\frac{\nu_{c1}}{\nu}\right)^2}$$ $\lambda := \frac{c0}{\nu}$ wavelength in air-filled section

$\lambda_g := \frac{2\cdot\pi}{\beta}$ $\beta = 398.642$ $\lambda = 0.015$

$\lambda_g = 0.016$ guide wavelength

$\frac{\lambda_g}{4} = 3.94\times10^{-3}$ 3.94mm from boundary