ENEE 408E Optical System Design 2003
Second Examination

Tuesday, November 25, 2003, 11:00 - 12:15 pm

ANSWER 3 QUESTIONS
(if more than 3 are answered best 3 will count)

(1) The transformed impedance of a transparent window with air ($n = 1$) on both sides is

$$Z_3'' = Z_2'\left(\frac{Z_1' \cos k_2 d' + jZ_2' \sin k_2 d'}{Z_2' \cos k_2 d' + jZ_1' \sin k_2 d'}\right).$$

Use this result to show that there is zero reflectance for P-waves at Brewster's angle. (5pts.)

The parameter d' is the effective thickness of the window.

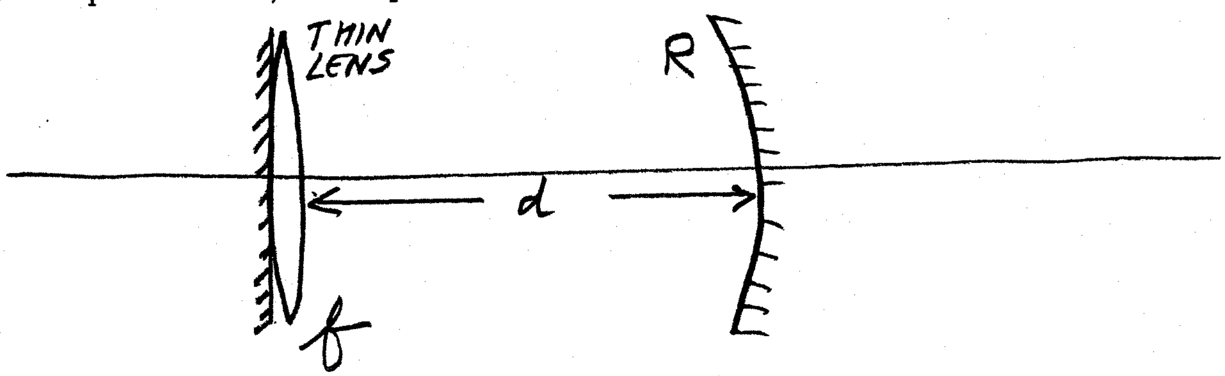
For P-waves $Z_1' = Z_0 \cos(\theta_1)$, where θ_1 is the angle of incidence. A window of thickness 2μm and refractive index 2 is illuminated with S-polarized *coherent* light of wavelength λ_0=1.3μm, at angle of incidence θ_1. All the incident light passes through the window. What is the angle of incidence? (This is not Brewster's angle) (5pts.)

Hint: For P-waves $Z' = Z_0 \cos\theta / n$.

Brewster's angle is $\theta_B = \arctan(n_2/n_1)$.

(2) Explain exactly what it means for an optical sequence to be *stable* for input paraxial rays. (1pt.)

Derive the stability condition for the resonator shown below, which contains a thin lens next to a plane mirror, and a spherical mirror of radius R. (5pts.)



If for the resonator above R=1m, f=0.5m, what is the maximum value of d for which the resonator is stable? (4pts.)

Hint: Sylvester's theorem provides a way of writing a $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ matrix raised to the *nth* power in terms of an angular factor ϕ for which $\cos\phi = (A + D)/2$.

(3) A graded index medium has

$$n(r) = n_0 \cos(\alpha r),$$

and it can be assumed that for all practical values of r for which a ray remains paraxial $\alpha r \ll 1$. Derive its ray transfer matrix. (6pts.)

For such a medium of length 10mm, n_0=1.46, and the index 1mm from the axis is 1.45. What is the focal length of the medium. (2pts.) What is the shortest length of medium (l >0) that will not change the ray parameters of an input wave? (2pts.)

Hint: The simplified equation for light rays is

$$\frac{d^2 r}{dz^2}\hat{\rho} = \frac{1}{n}\text{grad}n.$$

(4) The Gaussian beam parameter q is usually defined as

$$\frac{1}{q} = \frac{1}{R} - j\frac{\lambda}{\pi w^2}$$

and q varies linearly in a uniform medium according to $q(z) = q(z=0) + z$. Explain what q tells us about the characteristics of the Gaussian beam. (2pts.) If the beam waist of a Gaussian beam is at $z = 0$ prove that at distance z from the beamwaist

$$w^2(z) = w_0^2\left[1 + \left(\frac{\lambda z}{\pi w_0^2}\right)^2\right],$$

and

$$R(z) = z\left[1 + \left(\frac{\pi w_0^2}{\lambda z}\right)^2\right]. \quad \text{(4pts.)}$$

A Gaussian beam with w_0=1mm and λ_0=1.55μm has its beamwaist at the input plane of an optical system with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -20 & 41 \end{pmatrix}$$

Where is the beamwaist of the emerging Gaussian beam relative to the output plane of the system, and what is the new minimum spotsize? (4pts.) You can assume that the image and object spaces both have $n = 1$.

ENEE 408E 2003. Solutions to Questions on Second Examination

(1) At Brewster's angle $\tan(\theta_B) = \frac{n_2}{n_1}$ and consequently

$$\sin(\theta_B) = \frac{n_1}{\sqrt{n_1^2 + n_2^2}} \qquad \cos(\theta_B) = \frac{n_2}{\sqrt{n_1^2 + n_2^2}}$$

Substituting in the equation

$Z_2' = Z_0 \cos(\theta_2)/n_2$ gives $Z_2' = Z_1'$ and conseqently $Z_3'' = Z_1'$
There is then no reflection.

For the window to let all the light through its effective thickness must be an integral number of wavelengths, in other words

$$d \cdot \cos(\theta_2) = \frac{m \cdot \lambda_0}{n_2} \qquad d \cdot \sqrt{1 - \sin(\theta_2)^2} = m \cdot \frac{\lambda_0}{n_2}$$

and also $$d \cdot \sqrt{1 - \left(\frac{n_1 \cdot \sin(\theta 1)}{n_2}\right)^2} = m \cdot \frac{\lambda_0}{n_2}$$

$d := 2 \cdot 10^{-6}$ $\quad n_2 := 2$ $\quad \lambda_0 := 1.3 \cdot 10^{-6}$ $\quad n_1 := 1$ $\quad \frac{d \cdot n_2}{\lambda_0} = 3.077$ gives a clue

$m := 3$ Trial and error

$$\theta_1 := \operatorname{asin}\left[\frac{\left(n_2^2 \cdot d^2 - m^2 \cdot \lambda_0^2\right)^{\left(\frac{1}{2}\right)}}{(d \cdot n_1)}\right]$$

$$\frac{\theta_1}{deg} = 26.386$$

(2) A sequence is stable if

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 = \begin{pmatrix} An & Bn \\ Cn & Dn \end{pmatrix}$$

has elements An, Bn, Cn, and Dn that remain bounded for all values of n

The overall ray transfer matrix for the equvalent, "unfolded" sequence is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

which is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-2}{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} \frac{(-2 \cdot d + R)}{R} & 2 \cdot d \cdot \frac{(-d + R)}{R} \\ -2 \cdot \frac{(R - 2 \cdot d + f)}{(f \cdot R)} & \frac{\left(-4 \cdot d \cdot R + 4 \cdot d^2 - 2 \cdot d \cdot f + f \cdot R\right)}{(f \cdot R)} \end{bmatrix}$$

$$A = 1 - 2 \cdot \frac{d}{R} \qquad D = \frac{\left(-4 \cdot d \cdot R + 4 \cdot d^2 - 2 \cdot d \cdot f + f \cdot R\right)}{(f \cdot R)}$$

$$\cos\phi = \frac{1 - 2 \cdot \frac{d}{R} + \frac{\left(-4 \cdot d \cdot R + 4 \cdot d^2 - 2 \cdot d \cdot f + f \cdot R\right)}{(f \cdot R)}}{2}$$

$$\cos\phi = \frac{\left(f \cdot R - 2 \cdot d \cdot f - 2 \cdot d \cdot R + 2 \cdot d^2\right)}{(f \cdot R)}$$

The stability condition is

$$\left| 1 - 2 \cdot \frac{d}{R} - 2 \cdot \frac{d}{f} + \frac{2 \cdot d^2}{f \cdot R} \right|$$ less than or equal to 1

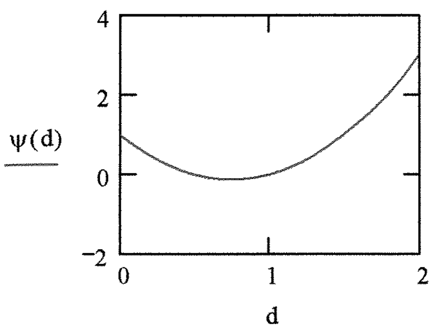
gives

$$0 \le \left(1-\frac{d}{f}\right)\cdot\left(1-\frac{d}{R}\right) \le 1$$

$R := 1 \qquad f := 0.5$

$d := 0, 0.01 .. 2$

$$\psi(d) := \left(1-\frac{d}{f}\right)\cdot\left(1-\frac{d}{R}\right)$$



$d := 1$

Given

$$\left(1-\frac{d}{f}\right)\cdot\left(1-\frac{d}{R}\right) = 1$$

$dmax := find(d)$

$dmax = 1.5$

Stable for $\quad 0 \le d \le 1.5$

(3) The differential equation for light rays gives

$$\frac{d^2 r}{dz^2} = -\alpha\cdot\tan(\alpha r)$$

For small values of αr, $\tan(\alpha r)=\alpha r$, so the solution is

$$r = A\cdot\sin(\alpha\cdot z) + B\cdot\cos(\alpha\cdot z)$$

$$\frac{dr}{dz} = A\cdot\alpha\cdot\cos(\alpha\cdot z) - B\cdot\alpha\cdot\sin(\alpha\cdot z)$$

Clearly, $B=r_{in}$, $A=r'_{in}/\alpha$

The ray transfer matrix is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \cos(\alpha\cdot z) & \frac{1}{\alpha}\cdot\sin(\alpha\cdot z) \\ -\alpha\cdot\sin(\alpha\cdot z) & \cos(\alpha\cdot z) \end{pmatrix}$$

$n_0 := 1.46 \qquad n_r := 1.45 \qquad r := 10^{-3} \quad z := 10\cdot 10^{-3}$

$n_r = n_0 \cdot \cos(\alpha \cdot r)$

$$\alpha := \frac{\operatorname{acos}\left(\frac{n_r}{n_0}\right)}{r}$$

$\alpha = 117.108$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} \cos(\alpha \cdot z) & \frac{1}{\alpha} \cdot \sin(\alpha \cdot z) \\ -\alpha \cdot \sin(\alpha \cdot z) & \cos(\alpha \cdot z) \end{pmatrix}$$

$$f := \frac{-1}{C}$$

$f = 9.27 \times 10^{-3}$ **this is the focal length**

For no change in ray parameters, matrix must be an indentity matrix

$\alpha \cdot z = 2 \cdot \pi$

$$z := \frac{2 \cdot \pi}{\alpha}$$

$z = 0.054$ **minimum length for no change**

(4) Derivation uses

$$q_0 = \frac{j \cdot \pi \cdot w_0^2}{\lambda_0}$$ **and $q=q_0+z$**

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} 1 & -2 \\ -20 & 41 \end{pmatrix}$$ **Ray transfer matrix of system, note that det( )=1**

$\frac{-1}{C} = 0.05$ $w_0 := 10^{-3}$ $\lambda_0 := 1.55 \cdot 10^{-6}$ **input parameters**

$$q_{in} := i \cdot \frac{\pi \cdot w_0^2}{\lambda_0}$$

$$q_{out} := \frac{A \cdot q_{in} + B}{C \cdot q_{in} + D}$$

$q_{out} = -0.049 + 6.097i \times 10^{-4}$ **Beamwaist is 49mm to right of output plane**

$$w_{new} := \sqrt{\frac{\lambda_0 \cdot \operatorname{Im}(q_{out})}{\pi}}$$ $w_{new} = 1.734 \times 10^{-5}$ **new minimum spotsize is 17.34μm**