ENEE 408E Design Problems #5 Solutions

(1) $R_1 := 10^{-1}$ $R_2 := -10^{-1}$ radii of curvature

$d := 20 \cdot 10^{-3}$ thickness

$n := 1.713$ ref. index

The ray transfer matrix for the thick lens is:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ \frac{n-1}{R_2} & n \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1-n}{n \cdot R_1} & \frac{1}{n} \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0.91675 & 0.01168 \\ -13.66646 & 0.91675 \end{pmatrix}$$

$$\left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right| = 1$$ Check on determinant

$$\frac{-1}{C} = 0.07317$$ focal length

(a) $w_0 := 5 \cdot 10^{-3}$ $\lambda_0 := 488 \cdot 10^{-9}$

$$q_{in} := i \cdot \frac{\pi \cdot w_0^2}{\lambda_0}$$

$$q_{out} := \frac{A \cdot q_{in} + B}{C \cdot q_{in} + D}$$

$$q_{out} = -0.06708 + 3.32673i \times 10^{-5}$$

New beamwaist is 67mm to right of output plane

$$w_{new} := \sqrt{\frac{\lambda_0 \cdot \mathrm{Im}(q_{out})}{\pi}}$$

$w_{new} = 2.27323 \times 10^{-6}$ new spotsize is 2.273μm

(b) **Include the additional 1m in the ray transfer matrix**

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ \frac{n-1}{R_2} & n \end{pmatrix} \cdot \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1-n}{n \cdot R_1} & \frac{1}{n} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right| = 1$$

$$w_0 := 0.1 \cdot 10^{-3} \qquad \lambda_0 := 1550 \cdot 10^{-9}$$

$$q_{in} := i \cdot \frac{\pi \cdot w_0^{\,2}}{\lambda_0}$$

$$q_{out} := \frac{A \cdot q_{in} + B}{C \cdot q_{in} + D}$$

$$q_{out} = -0.07282 + 1.24627i \times 10^{-4}$$

New beamwaist is 72.82mm to right of output plane

$$w_{new} := \sqrt{\frac{\lambda_0 \cdot \mathrm{Im}(q_{out})}{\pi}}$$

$w_{new} = 7.84147 \times 10^{-6}$ **new spotsize is 78.41μm**

Problem 3 (a)

Thick Lens

	RDY	THI	RMD	GLA	CCY	THC	GLC
OBJ:	INFINITY	0.010000			100	100	
STO:	100.00000	20.000000		LAK8_SCHOTT	100	100	
> 2:	-100.00000	66.221900			100	100	
IMG:	INFINITY	0.000000			100	100	

SPECIFICATION DATA

EPD	20.00000
DIM	MM
WL	488.00
REF	1
WTW	1
XAN	0.00000
YAN	0.00000
WTF	1.00000
VUY	0.00000
VLY	0.00000

REFRACTIVE INDICES

GLASS CODE	488.00
LAK8_SCHOTT	1.721995

Note that n is not exactly 1.73 since 488nm is not the reference wavelength

No solves defined in system

No pickups defined in system

INFINITE CONJUGATES

EFL	72.2832
BFL	66.2219
FFL	-66.2219
FNO	3.6142

AT USED CONJUGATES

RED	-1.0917
FNO	-0.5458
OBJ DIS	0.0100
TT	86.2319
IMG DIS	66.2219
OAL	20.0000
PARAXIAL IMAGE	
HT	0.0000

ANG 0.0000
ENTRANCE PUPIL
DIA 20.0000
THI 0.0000
EXIT PUPIL
DIA 21.7759
THI -12.8963

CODE V Output

WRX 0.1
WRY 0.1
RCX 10000000000000
RCY 10000000000000
GO

To see formatting of this output run Code V

GAUSSIAN BEAM PROPAGATION

Thick Lens POSITION 1

WAVELENGTH = 1550.0 NM DIMENSIONS = MILLIMETERS
FIELD POSITION = (0.00, 0.00)

BEAM WAVEFRONT RADIUS PHASE WAIST RADIUS
PROPAGATION BEAM RADIUS ORIENTATION OF CURVATURE
ORIENTATION BEFORE DISTANCE FROM
DISTANCE TO ON SURFACE (DEGREES) BEFORE REFRACTION
(DEGREES) REFRACTION WAIST TO SURFACE
SUR NEXT SURFACE X Y X Y X Y X Y

OBJ 1000.0000 0.1000 0.1000 0.0 INF INF 0.0 0.1000 0.1000 -
0.0000 -0.0000
1 20.0000 4.9348 4.9348 0.0 -1000.4108-1000.4108 0.0 0.1000 0.1000
1000.0000 1000.0000
2 75.6862 4.5908 4.5908 0.0 266.8814 266.8814 0.0 0.0170 0.0170 -
266.8778 -266.8778
IMG **0.0081** **0.0081** 0.0 -848.6113 -848.6113 0.0 0.0081
0.00810.21192e-40.21192e-4

Note new beamwaist minimum spotsize is 81 micrometers. Distance to beam waist is 75.682 mm.

(3)

$\lambda 0 := 780 \cdot 10^{-9}$ Wavelength

$n0 := 1.5$ reasonable value

GRIN lens needs to be about 3 mm in diameter so as to not truncate the output beam

$nr := 1.499178$ test value for index at edge of GRIN lens, keep adjusting until final answer is what you want.

$$nr = n0 - 0.5 \cdot n2 \cdot r^2$$

$$-2. \cdot \frac{(nr - 1. \cdot n0)}{r^2}$$ Symbolic solution for n2

$$n2 := -2. \cdot \frac{(nr - 1. \cdot n0)}{\left(1.5 \cdot 10^{-3}\right)^2}$$

$n2 = 730.667$ n2 value for GRIN lens

$d := 10 \cdot 10^{-2}$ Test length of medium

$w01 := 15 \cdot 10^{-6}$ input beamwaist

$w02 := 0.5 \cdot 10^{-3}$ desired output beamwaist

Setup ray transfer matrix

$$A := \cos\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$$

$$B := \sqrt{\frac{n0}{n2}} \cdot \sin\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$$

$$C := -\sqrt{\frac{n2}{n0}} \cdot \sin\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$$

$$D := \cos\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$$

$$qin := \frac{i \cdot \pi \cdot w01^2 \cdot n0}{\lambda 0}$$

$$qout := \frac{A \cdot qin + B}{C \cdot qin + D}$$

Choose A=D=0

$$\cos\left(\sqrt{\frac{n2}{n0}}\cdot d\right) = 0$$

Symbolic solution for d

$$\frac{1}{2}\cdot\frac{\pi}{\sqrt{n2}}\cdot\sqrt{n0}$$

$$d := \frac{1}{2}\cdot\frac{\pi}{\sqrt{n2}}\cdot\sqrt{n0}$$

$d = 0.071$ GRIN lens length

$$A := \cos\left(\sqrt{\frac{n2}{n0}}\cdot d\right)$$

$A = 0$

$$B := \sqrt{\frac{n0}{n2}}\cdot\sin\left(\sqrt{\frac{n2}{n0}}\cdot d\right)$$

$B = 0.045$

$$C := -\sqrt{\frac{n2}{n0}}\cdot\sin\left(\sqrt{\frac{n2}{n0}}\cdot d\right)$$

$C = -22.071$

$$D := \cos\left(\sqrt{\frac{n2}{n0}}\cdot d\right)$$

$D = 0$

$$qin := \frac{i\cdot\pi\cdot w01^{2}\cdot n0}{\lambda 0}$$

$$qout := \frac{A\cdot qin + B}{C\cdot qin + D}$$

$qout = 1.51i$

note that qout is pure imaginary so the beamwaist is on the output face

$$qout = \frac{i\cdot\pi\cdot w02^{2}\cdot n0}{\lambda 0}$$

$$w02 := \sqrt{\frac{\lambda 0\cdot qout}{i\cdot\pi\cdot n0}}$$

$w02 = 5\times 10^{-4}$

Final beam diameter is what is required

(4) $\nu 0 := 10\cdot 10^{-3}$

$\lambda 0 := 880\cdot 10^{-9}$

$n0 := 1.5$

$n2 := 0.03$

$d := 1$

$$qin := \frac{i \cdot \pi \cdot w0^2 \cdot n0}{\lambda 0}$$

Beam parameter just inside GRIN lens

Setup ray transfer matrix for GRIN medium

$$A := \cos\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$$

$$B := \sqrt{\frac{n0}{n2}} \cdot \sin\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$$

$$C := -\sqrt{\frac{n2}{n0}} \cdot \sin\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$$

$$D := \cos\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$$

Ray transfer matrix of output face of GRIN

$a := 1$

$b := 0$

$c := 0$

$d := n0$

Ray transfer matrix for uniform length of 1m

$A1 := 1$

$B1 := 1$

$C1 := 0$

$D1 := 1$

Overall ray transfer matrix

$$\begin{pmatrix} a1 & b1 \\ c1 & d1 \end{pmatrix} := \begin{pmatrix} A1 & B1 \\ C1 & D1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$qout := \frac{a1 \cdot qin + b1}{c1 \cdot qin + d1}$$

$qout = -31.823 + 3.106i$

$\frac{1}{qout} = -0.031 - 3.039i \times 10^{-3}$

Remember that $\frac{1}{qout} = \frac{1}{R} - i \cdot \frac{\lambda 0}{\pi \cdot w^2}$

$$R := \frac{1}{Re\left(\frac{1}{qout}\right)}$$

$R = -32.126$ Radius of curvature

$$w := \frac{\lambda 0}{\pi \cdot Im\left(\frac{1}{qout}\right)}$$

$w = 9.218 \times 10^{-5}$ Spot size

(5) $R1 := 1.5$ $R2 := 3$ $d := 1$ $\lambda 0 := 488 \cdot 10^{-9}$

$$R1 = t1 \cdot \left[1 + \left[\pi \cdot w0 \cdot \frac{w0}{(\lambda 0 \cdot t1)}\right]^2\right]$$

t1 is distance of beamwaist from R1

Mathcad symbolic calculation of two possible values for t1

$$\begin{bmatrix} \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) \\ \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) \end{bmatrix}$$

$$t11 = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$t12 = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$R2 = t2 \cdot \left[1 + \left[\pi \cdot w0 \cdot \frac{w0}{(\lambda 0 \cdot t2)}\right]^2\right]$$

t2 is distance of beamwaist from R2

Symbolic Mathcad evaluation of two possible values for t2

$$\begin{bmatrix} \frac{-1}{\left(2\cdot\lambda0^2\right)}\cdot\left(-R2\cdot\lambda0^2+\lambda0\cdot\sqrt{R2^2\cdot\lambda0^2-4\cdot\pi^2\cdot w0^4}\right) \\ \frac{-1}{\left(2\cdot\lambda0^2\right)}\cdot\left(-R2\cdot\lambda0^2-\lambda0\cdot\sqrt{R2^2\cdot\lambda0^2-4\cdot\pi^2\cdot w0^4}\right) \end{bmatrix}$$

$$t21=\frac{-1}{\left(2\cdot\lambda0^2\right)}\cdot\left(-R2\cdot\lambda0^2+\lambda0\cdot\sqrt{R2^2\cdot\lambda0^2-4\cdot\pi^2\cdot w0^4}\right)$$

$$t22=\frac{-1}{\left(2\cdot\lambda0^2\right)}\cdot\left(-R2\cdot\lambda0^2-\lambda0\cdot\sqrt{R2^2\cdot\lambda0^2-4\cdot\pi^2\cdot w0^4}\right)$$

$$d=t11+t21$$

Symbolic substitution

$$d=t11-\frac{1}{\left(2\cdot\lambda0^2\right)}\cdot\left(-R2\cdot\lambda0^2+\lambda0\cdot\sqrt{R2^2\cdot\lambda0^2-4\cdot\pi^2\cdot w0^4}\right)$$

$$d=\frac{-1}{\left(2\cdot\lambda0^2\right)}\cdot\left(-R1\cdot\lambda0^2+\lambda0\cdot\sqrt{R1^2\cdot\lambda0^2-4\cdot\pi^2\cdot w0^4}\right)-\frac{1}{\left(2\cdot\lambda0^2\right)}\cdot\left(-R2\cdot\lambda0^2+\lambda0\cdot\sqrt{R2^2\cdot\lambda0^2-4\cdot\pi^2\cdot w0^4}\right)$$

Calculate w0 symbolically. There are 4 possible solutions.

$$\begin{bmatrix} \left[-\lambda0^2\cdot d\cdot\frac{\left(R2^2\cdot d-R2^2\cdot R1+d^3+3\cdot d\cdot R1\cdot R2-R1^2\cdot R2-2\cdot d^2\cdot R1-2\cdot d^2\cdot R2+d\cdot R1^2\right)}{\left[(2\cdot d-R1-R2)^2\cdot\pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ i\cdot\left[-\lambda0^2\cdot d\cdot\frac{\left(R2^2\cdot d-R2^2\cdot R1+d^3+3\cdot d\cdot R1\cdot R2-R1^2\cdot R2-2\cdot d^2\cdot R1-2\cdot d^2\cdot R2+d\cdot R1^2\right)}{\left[(2\cdot d-R1-R2)^2\cdot\pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -\left[-\lambda0^2\cdot d\cdot\frac{\left(R2^2\cdot d-R2^2\cdot R1+d^3+3\cdot d\cdot R1\cdot R2-R1^2\cdot R2-2\cdot d^2\cdot R1-2\cdot d^2\cdot R2+d\cdot R1^2\right)}{\left[(2\cdot d-R1-R2)^2\cdot\pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -i\cdot\left[-\lambda0^2\cdot d\cdot\frac{\left(R2^2\cdot d-R2^2\cdot R1+d^3+3\cdot d\cdot R1\cdot R2-R1^2\cdot R2-2\cdot d^2\cdot R1-2\cdot d^2\cdot R2+d\cdot R1^2\right)}{\left[(2\cdot d-R1-R2)^2\cdot\pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \end{bmatrix}$$

$$w0 := \left[-\lambda0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)}$$

$$w0 = 3.409 \times 10^{-4}$$

$$t11 := \frac{-1}{\left(2 \cdot \lambda0^2\right)} \cdot \left(-R1 \cdot \lambda0^2 + \lambda0 \cdot \sqrt{R1^2 \cdot \lambda0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$t11 = 0.7$$

$$t21 := \frac{-1}{\left(2 \cdot \lambda0^2\right)} \cdot \left(-R2 \cdot \lambda0^2 + \lambda0 \cdot \sqrt{R2^2 \cdot \lambda0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$t21 = 0.2$$

This is not a satisfactory solution. Try another combination

$$d = t12 + t22$$

$$d = \frac{-1}{\left(2 \cdot \lambda0^2\right)} \cdot \left(-R1 \cdot \lambda0^2 - \lambda0 \cdot \sqrt{R1^2 \cdot \lambda0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) + t22$$

$$d = \frac{-1}{\left(2 \cdot \lambda0^2\right)} \cdot \left(-R1 \cdot \lambda0^2 - \lambda0 \cdot \sqrt{R1^2 \cdot \lambda0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) - \frac{1}{\left(2 \cdot \lambda0^2\right)} \cdot \left(-R2 \cdot \lambda0^2 - \lambda0 \cdot \sqrt{R2^2 \cdot \lambda0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$\begin{bmatrix} \left[-\lambda0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ i \cdot \left[-\lambda0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -\left[-\lambda0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -i \cdot \left[-\lambda0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \end{bmatrix}$$

$$w0 := \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)}$$

$w0 = 3.409 \times 10^{-4}$ This is the minimum spotsize

$$t12 := \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) \qquad t12 = 0.8$$

$$t22 := \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) \qquad t22 = 2.8$$

Not satisfactory. Try another combination.

$d = t12 + t21$

$$d = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) + t21$$

$$d = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) - \frac{1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$\begin{bmatrix} \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ i \cdot \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -\left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -i \cdot \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \end{bmatrix}$$

$$w0 := \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)}$$

$w0 = 3.409 \times 10^{-4}$ Spot size comes out the same

$t12 = 0.8$

$t21 = 0.2$

$t11 = 0.7$

$t22 = 2.8$

Only the solution t1=0.0.8, t2=0.2 satisfies t1+t2=d

(6) Vary d, the length of the GRIN lens to get the correct answer -- trial and error

$\lambda 0 := 1.55 \cdot 10^{-6}$

$w0 := 0.5 \cdot 10^{-3}$

$qin := \frac{i \cdot \pi \cdot w0^2}{\lambda 0}$ Beam parameter at original beamwaist

Input beam divergence is

$$\theta := \frac{\lambda 0}{\pi \cdot w0}$$

For a focused spot size of 10 micrometers an approximate solution can be found from

$$f := \frac{10 \cdot 10^{-6}}{\theta}$$

$f = 0.01$ Use this as a starting value

$n0 := 1.5$

Choose a GRIN lens radius of 1mm -- will not truncate input beam

$nr = n0 - 0.5 \cdot n2 \cdot r^2$ Symbolic solution for n2

$$-2. \cdot \frac{(nr - 1. \cdot n0)}{r^2}$$

$nr := 1.485$ Initial guess

$r := 10^{-3}$

$n2 := -2. \cdot \frac{(nr - 1. \cdot n0)}{r^2}$

$\frac{-1}{f} = -\sqrt{\frac{n2}{n0}} \cdot \sin\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$ focal length as a function of GRIN lens parameters

$$\frac{\operatorname{asin}\left[\frac{\sqrt{n0}}{(\sqrt{n2} \cdot f)}\right]}{\sqrt{n2}} \cdot \sqrt{n0}$$

Symbolic solution for d that will give a particular focal length

$$d := \frac{\operatorname{asin}\left[\frac{\sqrt{n0}}{(\sqrt{n2} \cdot f)}\right]}{\sqrt{n2}} \cdot \sqrt{n0}$$

$d = 5.461 \times 10^{-3}$ Use this as a starting value

$d := 0.00364$ This length found by trial and error to focus beam to 10 micrometer spotsize

Ray transfer matrix for initial 10mm

$a2 := 1$

$b2 := 10^{-2}$

$c2 := 0$

$d2 := 1$

Ray transfer matrix for front face of GRIN

$a3 := 1$

$b3 := 0$

$c3 := 0$

$d3 := \frac{1}{n0}$

Setup ray transfer matrix for GRIN medium

$A := \cos\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$

$B := \sqrt{\frac{n0}{n2}} \cdot \sin\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$

$$C := -\sqrt{\frac{n2}{n0}} \cdot \sin\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$$

$$D := \cos\left(\sqrt{\frac{n2}{n0}} \cdot d\right)$$

$-\frac{1}{C} = 0.014$ focal length of GRIN

Ray transfer matrix of output face of GRIN

$a := 1$

$b := 0$

$c := 0$

$d := n0$

Overall ray transfer to just outside output face of GRIN

$$\begin{pmatrix} a4 & b4 \\ c4 & d4 \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} a3 & b3 \\ c3 & d3 \end{pmatrix} \cdot \begin{pmatrix} a2 & b2 \\ c2 & d2 \end{pmatrix}$$

$$qout := \frac{a4 \cdot qin + B}{c4 \cdot qin + D}$$

$qout = -8.331 \times 10^{-3} + 2.028i \times 10^{-4}$

Distance to focus is df

$df := -Re(qout)$

$df = 8.331 \times 10^{-3}$

$$wfocus := \sqrt{\frac{\lambda 0 \cdot Im(qout)}{\pi}}$$

$wfocus = 1 \times 10^{-5}$ value desired

(7) $R1 := 4$ $R2 := 2$ $d := 0.5$ $\lambda 0 := 1.06 \cdot 10^{-6}$

$$R1 = t1 \cdot \left[1 + \left[\pi \cdot w0 \cdot \frac{w0}{(\lambda 0 \cdot t1)}\right]^2\right]$$

t1 is distance of beamwaist from R1

Mathcad symbolic calculation of two possible values for t1

$$\begin{bmatrix} \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) \\ \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) \end{bmatrix}$$

$$t11 = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$t12 = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$R2 = t2 \cdot \left[1 + \left[\pi \cdot w0 \cdot \frac{w0}{(\lambda 0 \cdot t2)}\right]^2\right]$$

t2 is distance of beamwaist from R2

Symbolic Mathcad evaluation of two possible values for t2

$$\begin{bmatrix} \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) \\ \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) \end{bmatrix}$$

$$t21 = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$t22 = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$d = t11 + t21$$

Symbolic substitution

$$d = t11 - \frac{1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$d = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) - \frac{1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

Calculate w0 symbolically. There are 4 possible solutions.

$$\begin{bmatrix} \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ i \cdot \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -\left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -i \cdot \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \end{bmatrix}$$

Choose physically reasonable solution

$$w0 := \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)}$$

$w0 = 5.064 \times 10^{-4}$ This is the spotsize required

$$t11 := \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$t11 = 0.15$

$$t21 := \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 + \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$t21 = 0.35$

This is a satisfactory solution

Test another combination

$d = t12 + t22$

$$d = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) + t22$$

$$d = \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) - \frac{1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right)$$

$$\begin{bmatrix} \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ i \cdot \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -\left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -i \cdot \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \end{bmatrix}$$

$$w0 := \left[-\lambda 0^2 \cdot d \cdot \frac{\left(R2^2 \cdot d - R2^2 \cdot R1 + d^3 + 3 \cdot d \cdot R1 \cdot R2 - R1^2 \cdot R2 - 2 \cdot d^2 \cdot R1 - 2 \cdot d^2 \cdot R2 + d \cdot R1^2\right)}{\left[(2 \cdot d - R1 - R2)^2 \cdot \pi^2\right]}\right]^{\left(\frac{1}{4}\right)}$$

$w0 = 5.064 \times 10^{-4}$ Same spotsize

$$t12 := \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R1 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R1^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) \qquad t12 = 3.85$$

$$t22 := \frac{-1}{\left(2 \cdot \lambda 0^2\right)} \cdot \left(-R2 \cdot \lambda 0^2 - \lambda 0 \cdot \sqrt{R2^2 \cdot \lambda 0^2 - 4 \cdot \pi^2 \cdot w0^4}\right) \qquad t22 = 1.65$$

Not satisfactory. Try another combination.

$d = t12 + t21$

$$d = \frac{-1}{\left(2\cdot\lambda0^2\right)}\cdot\left(-R1\cdot\lambda0^2 - \lambda0\cdot\sqrt{R1^2\cdot\lambda0^2 - 4\cdot\pi^2\cdot w0^4}\right) + t21$$

$$d = \frac{-1}{\left(2\cdot\lambda0^2\right)}\cdot\left(-R1\cdot\lambda0^2 - \lambda0\cdot\sqrt{R1^2\cdot\lambda0^2 - 4\cdot\pi^2\cdot w0^4}\right) - \frac{1}{\left(2\cdot\lambda0^2\right)}\cdot\left(-R2\cdot\lambda0^2 - \lambda0\cdot\sqrt{R2^2\cdot\lambda0^2 - 4\cdot\pi^2\cdot w0^4}\right)$$

$$\begin{bmatrix} \left[-\lambda0^2\cdot d\cdot\frac{\left(R2^2\cdot d - R2^2\cdot R1 + d^3 + 3\cdot d\cdot R1\cdot R2 - R1^2\cdot R2 - 2\cdot d^2\cdot R1 - 2\cdot d^2\cdot R2 + d\cdot R1^2\right)}{\left[(2\cdot d - R1 - R2)^2\cdot\pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ i\cdot\left[-\lambda0^2\cdot d\cdot\frac{\left(R2^2\cdot d - R2^2\cdot R1 + d^3 + 3\cdot d\cdot R1\cdot R2 - R1^2\cdot R2 - 2\cdot d^2\cdot R1 - 2\cdot d^2\cdot R2 + d\cdot R1^2\right)}{\left[(2\cdot d - R1 - R2)^2\cdot\pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -\left[-\lambda0^2\cdot d\cdot\frac{\left(R2^2\cdot d - R2^2\cdot R1 + d^3 + 3\cdot d\cdot R1\cdot R2 - R1^2\cdot R2 - 2\cdot d^2\cdot R1 - 2\cdot d^2\cdot R2 + d\cdot R1^2\right)}{\left[(2\cdot d - R1 - R2)^2\cdot\pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \\ -i\cdot\left[-\lambda0^2\cdot d\cdot\frac{\left(R2^2\cdot d - R2^2\cdot R1 + d^3 + 3\cdot d\cdot R1\cdot R2 - R1^2\cdot R2 - 2\cdot d^2\cdot R1 - 2\cdot d^2\cdot R2 + d\cdot R1^2\right)}{\left[(2\cdot d - R1 - R2)^2\cdot\pi^2\right]}\right]^{\left(\frac{1}{4}\right)} \end{bmatrix}$$

$$w0 := \left[-\lambda0^2\cdot d\cdot\frac{\left(R2^2\cdot d - R2^2\cdot R1 + d^3 + 3\cdot d\cdot R1\cdot R2 - R1^2\cdot R2 - 2\cdot d^2\cdot R1 - 2\cdot d^2\cdot R2 + d\cdot R1^2\right)}{\left[(2\cdot d - R1 - R2)^2\cdot\pi^2\right]}\right]^{\left(\frac{1}{4}\right)}$$

$w0 = 5.064 \times 10^{-4}$ Spot size comes out the same

$t12 = 3.85$

$t21 = 0.35$

$t11 = 0.15$

$t22 = 1.65$

Only the solution t1=0.15, t2=0.35 satisfies t1+t2=d

(8) $w0 := 0.2 \cdot 10^{-3}$

$\lambda 0 := 1.3 \cdot 10^{-6}$

$d := 200 \cdot 10^{-3}$

$$R := \frac{d}{2} \cdot \left[1 + \left[2 \cdot \pi \cdot w0 \cdot \frac{w0}{(\lambda 0 \cdot d)}\right]^2\right]$$

$R = 0.193$ meters