(2.1)

$\gamma_0(\nu_0) = 0.6\ m^{-1}$ $\qquad l = 3\ m$

$I_s(\nu_0) = 3\ W\ m^{-2}$

$I_0 = 2.5\ W$

(a) Input at line center, no saturation included

$I = I_0 e^{\gamma_0 l} = 2.5 e^{1.8}$ $= \underline{15.124\ W\ m^{-2}}$

(b) Input ~ 1 FWHM from line center

$$\gamma(\nu) = \frac{\gamma_0(\nu_0)}{1 + \left[\frac{2(\nu - \nu_0)}{\Delta\nu}\right]^2} = \frac{\gamma_0(\nu_0)}{5}$$

ie $\gamma(\nu) = 0.12\ m^{-1}$

$I = I_0 e^{0.36} = 2.5 e^{0.36}$ $= \underline{3.5833\ W\ m^{-2}}$

(c)

$$I = I_0 e^{\gamma_0(\nu_0) l - (I - I_0) I_s} \qquad (A)$$

I lies somewhere between 2.5 & 15.124 W m^{-2}

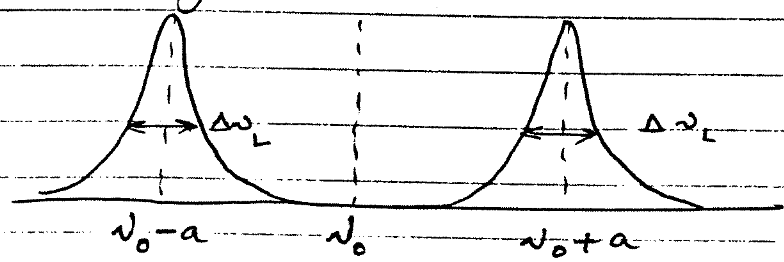
Mathcad gives $I = \underline{5.522\ W\ m^{-2}}$

(d) 1 FWHM from line center $\gamma_0 = 0.12\ m^{-1}$

$I_s(\nu) \propto \frac{1}{g(\nu_0, \nu)}$ so $I_s(\nu) = 15\ W\ m^{-2}$

Solve Eq (A) again $\qquad I = \underline{3.379\ W\ m^{-2}}$

(2.2) Overall gain profile looks like, for example:



Overall gain is $\propto \dfrac{1}{1+\left[\dfrac{2(\nu-\nu_0-a)}{\Delta\nu_L}\right]^2} + \dfrac{1}{1+\left[\dfrac{2(\nu-\nu_0+a)}{\Delta\nu_L}\right]^2}$

ie $\gamma(\nu) \propto \dfrac{1}{\Delta\nu_L^2 + 4(\nu-\nu_0-a)^2} + \dfrac{1}{(\Delta\nu_L)^2 + 4(\nu-\nu_0+a)^2}$

Gain of amplifier is a maximum where $d\gamma/d\nu = 0$

$$\frac{d\gamma}{d\nu} = \frac{-8(\nu-\nu_0-a)}{\left[\Delta\nu_L^2 + 4(\nu-\nu_0-a)^2\right]^2} - \frac{8(\nu-\nu_0+a)}{\left[\Delta\nu_L^2 + 4(\nu-\nu_0+a)^2\right]^2}$$

$\frac{d\gamma}{d\nu} = 0$ when $(\nu-\nu_0-a)\left[\Delta\nu_L^2 + 4(\nu-\nu_0+a)^2\right]^2 + (\nu-\nu_0+a)\left[\Delta\nu_L^2 + 4(\nu-\nu_0-a)^2\right]^2 = 0$

Write $\nu = \nu_0 + b$ ie $\nu - \nu_0 = b$

$$(b-a)\left[\Delta\nu_L^2 + 4(b+a)^2\right]^2 + (b+a)\left[\Delta\nu_L^2 + 4(b-a)^2\right]^2 = 0 \quad (1)$$

Case (1) $a \gg \Delta\nu_L$. In this case $b \approx a + \Delta$ where Δ is v. small

ie $\Delta\left[\Delta\nu_L^2 + 4(2a+\Delta)^2\right]^2 + (2a+\Delta)\left[\Delta\nu_L^2 + 4\Delta^2\right]^2 = 0$

Neglect terms in $\Delta^2, \Delta^3, \Delta^4$

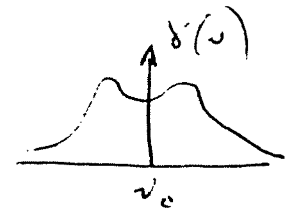
$$\Delta\left[(\Delta\nu_L)^2 + 16a^2\right]^2 + (2a+\Delta)(\Delta\nu_L)^4 = 0$$

$$\therefore \Delta\left\{\left[\Delta\nu_L^2 + 16a^2\right]^2 + \Delta\nu_L^4\right\} = -2a(\Delta\nu_L)^4$$

$$\Delta = \frac{-2a(\Delta\nu_L)^4}{\left[\Delta\nu_L^2 + 16a^2\right]^2 + \Delta\nu_L^4}$$

and for $a >> \Delta\nu_L$ $\quad \Delta = \dfrac{-2a(\Delta\nu_L)^4}{256\,a^4} = -\dfrac{2}{a^3}\left(\dfrac{\Delta\nu_L}{4}\right)^4$

Thus $b \simeq a - \dfrac{2}{a^3}\left(\dfrac{\Delta\nu_L}{4}\right)^4$



Case (ii)

For $a \sim \Delta\nu_L$, Solve Eq. ① for b

Case (iii) $a << \Delta\nu_L$. In this case $b = \Delta$ where Δ ~ v. small

Substitute in Eq. ①

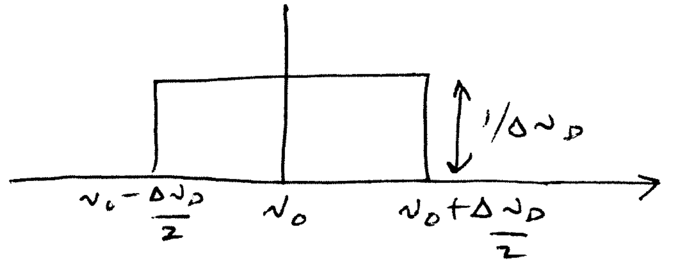
$$(\Delta - a)\left[\Delta\nu_L^2 + 4(\Delta+a)^2\right]^2 + (\Delta+a)\left[\Delta\nu_L^2 + 4(\Delta-a)^2\right]^2 = 0$$

Neglect small terms

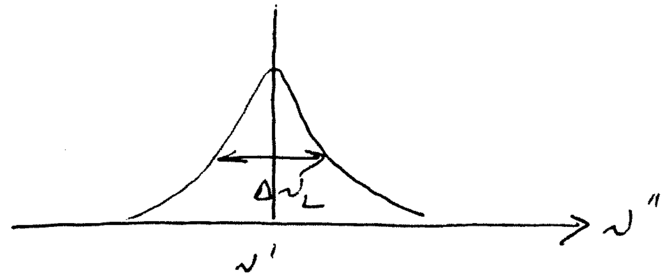
$$(\Delta - a)\,\Delta\nu_L^4 + (\Delta+a)\,(\Delta\nu_L)^4 = 0$$

$\therefore$ To first order $\Delta = 0$, ie $b = 0$

(2.3) $g_D(\nu_0, \nu)$



$g_L(\nu', \nu'')$



overall lineshape is

$$g_T(\nu_0, \nu) = \frac{1}{\Delta\nu_D} \int_{\nu_0 - \frac{\Delta\nu_D}{2}}^{\nu_0 + \frac{\Delta\nu_D}{2}} \frac{2}{\pi\Delta\nu_L} \frac{d\nu''}{1 + \left[\frac{2(\nu - \nu'')}{\Delta\nu_L}\right]^2}$$

Put $\frac{2(\nu - \nu'')}{\Delta\nu_L} = x \qquad d\nu'' = -\frac{\Delta\nu_L}{2}\, dx$

$$g_T(\nu_0, \nu) = \frac{-1}{\pi\Delta\nu_D} \int_{\frac{2}{\Delta\nu_L}\left(\nu - \nu_0 + \frac{\Delta\nu_D}{2}\right)}^{\frac{2}{\Delta\nu_L}\left(\nu - \nu_0 - \frac{\Delta\nu_D}{2}\right)} \frac{dx}{1 + x^2}$$

$$= \frac{1}{\pi\Delta\nu_D} \arctan x \Big|_{\frac{2}{\Delta\nu_L}\left(\nu - \nu_0 - \frac{\Delta\nu_D}{2}\right)}^{\frac{2}{\Delta\nu_L}\left(\nu - \nu_0 + \frac{\Delta\nu_D}{2}\right)}$$

$$g_T(\nu_0, \nu) = \frac{1}{\pi\Delta\nu_D}\left[\arctan \frac{2}{\Delta\nu_L}\left(\nu - \nu_0 + \frac{\Delta\nu_D}{2}\right) - \arctan \frac{2}{\Delta\nu_L}\left(\nu - \nu_0 - \frac{\Delta\nu_D}{2}\right)\right]$$

See following Mathcad plots

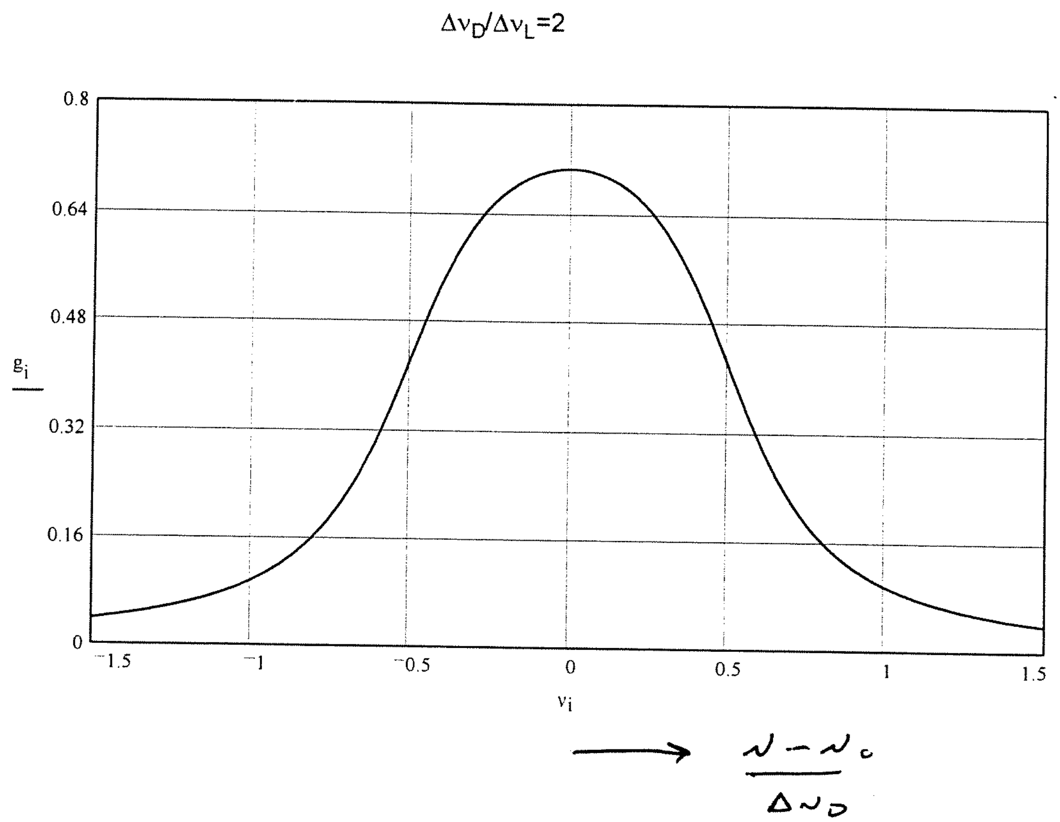
$i := 1, 2 .. 1001$

$\Delta\nu D := 1$

$\Delta\nu L := 0.5$

$$\nu_i := -1.5\cdot\Delta\nu D + \frac{i-1}{1000}\cdot 3\cdot\Delta\nu D$$

$$g_i := \frac{1}{\pi\cdot\Delta\nu D}\cdot\left[\operatorname{atan}\left[\frac{2}{\Delta\nu L}\cdot\left(\frac{\Delta\nu D}{2}+\nu_i\right)\right]-\operatorname{atan}\left[\frac{2}{\Delta\nu L}\cdot\left(\nu_i-\frac{\Delta\nu D}{2}\right)\right]\right]$$



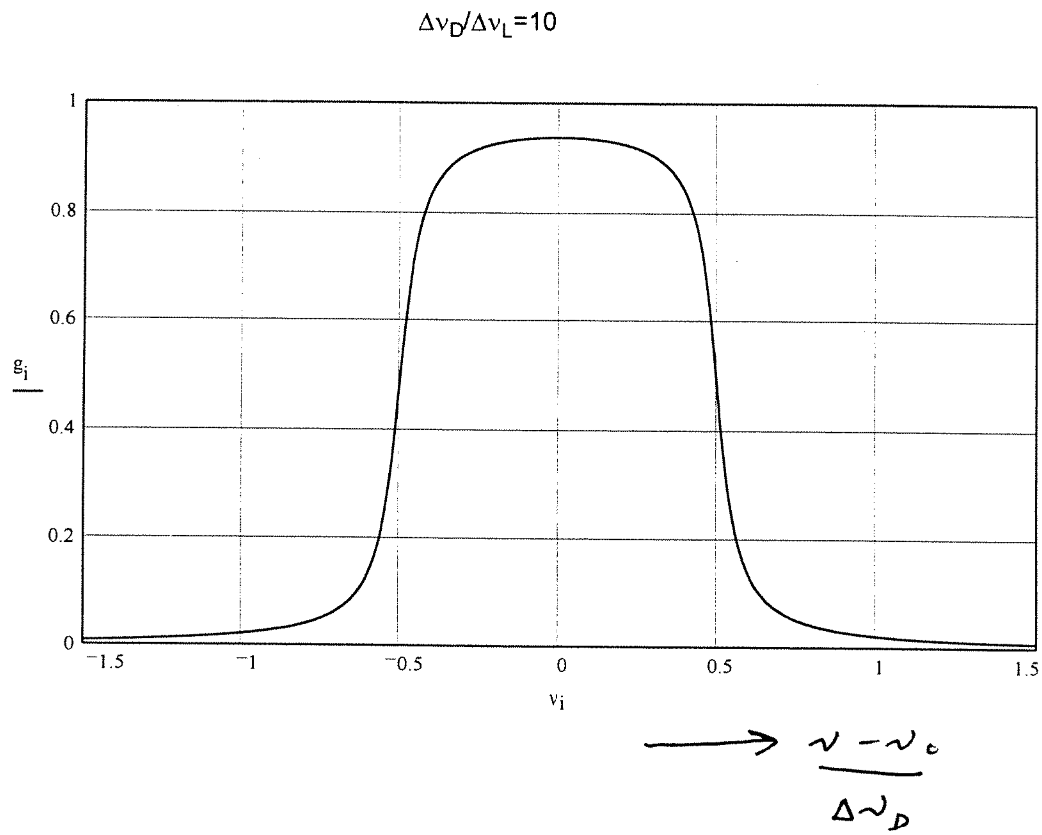
$i := 1, 2 .. 1001$

$\Delta\nu D := 1$

$\Delta\nu L := 0.1$

$\nu_i := -1.5\cdot\Delta\nu D + \frac{i-1}{1000}\cdot 3\cdot\Delta\nu D$

$$g_i := \frac{1}{\pi\cdot\Delta\nu D}\cdot\left[\operatorname{atan}\left[\frac{2}{\Delta\nu L}\cdot\left(\frac{\Delta\nu D}{2}+\nu_i\right)\right] - \operatorname{atan}\left[\frac{2}{\Delta\nu L}\cdot\left(\nu_i - \frac{\Delta\nu D}{2}\right)\right]\right]$$



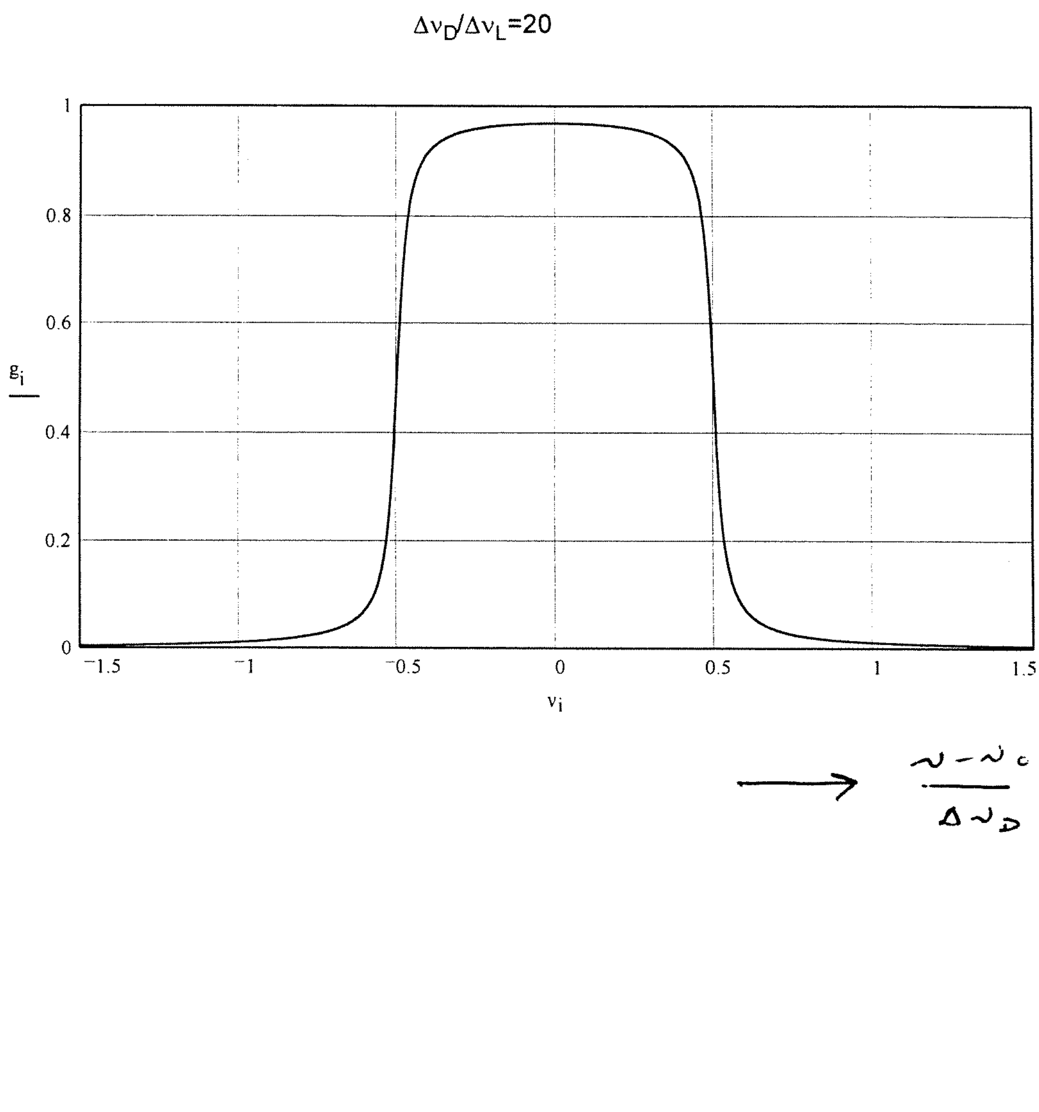
$i := 1, 2 .. 1001$

$\Delta\nu D := 1$

$\Delta\nu L := 0.05$

$$\nu_i := -1.5\cdot\Delta\nu D + \frac{i-1}{1000}\cdot 3\cdot\Delta\nu D$$

$$g_i := \frac{1}{\pi\cdot\Delta\nu D}\cdot\left[\operatorname{atan}\left[\frac{2}{\Delta\nu L}\cdot\left(\frac{\Delta\nu D}{2}+\nu_i\right)\right]-\operatorname{atan}\left[\frac{2}{\Delta\nu L}\cdot\left(\nu_i-\frac{\Delta\nu D}{2}\right)\right]\right]$$



(2.4) The stimulated emission rate is

$$W_{21}(\omega) = \frac{c^3 A_{21}}{8\pi h \omega^3}\, g(\omega_0,\omega)\, \rho(\omega) \qquad \sec^{-1}\ Hz^{-1}$$

For "white" input radiation

$$W_{21} = \int W_{21}(\omega)\, d\omega = \frac{c^3 A_{21}}{8\pi h\omega^3}\, \rho(\omega) \int_{-\infty}^{\infty} g(\omega_0,\omega)\, d\omega$$

$$= \frac{c^3 A_{21}}{8\pi h \omega^3}\, \rho(\omega)$$

$$= \frac{\lambda^3 A_{21}}{8\pi h}\, \rho(\omega)$$

For example given

$$W_{21} = \frac{(3\times10^8)^3\ 10^8 \times 1}{8\pi \times 6.6\times10^{-34} \times 10^{+2}}$$

$$= 1.6277\times10^{23}\ s^{-1}$$

For monochromatic input

$$W_{21} = \int W_{21}(\omega)\, d\omega = \frac{c^3 A_{21}}{8\pi h\omega^3}\, g(\omega_0,\omega)\, \rho_\omega$$

$$\rho = \frac{I}{c} \qquad \therefore W_{21} = \frac{c^2 A_{21}}{8\pi h\omega^3}\, I\, g(\omega_0,\omega)$$

In the example given $\quad g(\omega_0,\omega) = \dfrac{(2/\pi\Delta\omega)}{5}$

$$\therefore W = \frac{c^2 A_{21}}{8\pi h\omega^3}\, I \left(\frac{2}{5\pi\Delta\omega}\right) = \frac{(3\times10^8)^2 \times 10^8 \times 1}{20\pi^2 \times 6.6\times10^{-34} \times 10^{+2} \times 10^9}$$

$$= 6.9\times10^{4}\ s^{-1}$$

(2.5) $\Delta\nu = 10^9$ Hz

$\gamma_0(\nu_0) = 1\ m^{-1}$

$I_s(\nu_0) = 1\ W\ m^{-2}$

$\ell = 0.5\ m$

$I_0 = 2\ W\ m^{-2}$

(a) 500 MHz from line center $\gamma_c = 0.5\ m^{-1}$, $I_s = 2\ W\ m^{-2}$
w/o saturation $I_{out} = 2e^{0.25} = 2.568\ W\ m^{-2}$

(b) With saturation by Mathcad (see problem 2.1)
$I_{out} = 2.258\ W\ m^{-2}$

(c) At line center $\gamma_0 = 1\ m^{-1}$, $I_s = 1\ W\ m^{-2}$
with saturation by mathcad (see problem 2.1)

$I_{out} = 2.392\ W\ m^{-2}$

(d) The 2 signals above and below line center see the same saturation intensity
Intensity of radiation above line center is I_1
Intensity of radiation below line center is I_2

$$\frac{1}{I_1}\frac{dI_1}{dz} = \frac{\gamma_0}{1+(I_1+I_2)I_s} \quad \text{(i)}$$

Write $I = I_1 + I_2$

$$\frac{1}{I_2}\frac{dI_2}{dz} = \frac{\gamma_0}{1+(I_1+I_2)I_s} \quad \text{(ii)}$$

gives $$\frac{1}{I}\frac{dI}{dz} = \frac{\gamma_0}{(1+I/I_s)}$$

Substitute in (i)

$$\frac{1}{I_1}\frac{dI_1}{dz} = \frac{\gamma_0}{1 + (I_1^\circ + I_2^\circ)e^{\gamma_0 l - (I_1 + I_2 - I_1^\circ - I_2^\circ)/I_s}}$$

Substitute in (ii)

$$\frac{1}{I_2}\frac{dI_2}{dz} = \frac{\gamma_0}{1 + (I_1^\circ + I_2^\circ)e^{\gamma_0 l - (I_1 + I_2 - I_1^\circ - I_2^\circ)/I_s}}$$

where I_1° & I_2° are the input intensities

This is the exact solution and would need to be solved numerically with $I_1 = 2\ \mathrm{W\,m^{-2}}$, $I_2 = 1\ \mathrm{W\,m^{-2}}$, $I_s = 2\ \mathrm{W\,m^{-2}}$

(2.6) Saturation Intensity is $I_s(\nu) = \dfrac{8\pi h \nu^3}{c^2 \phi\, g(\nu_0, \nu)}$

$$\phi = A_{21}\tau_2\left[1 + (1 - A_{21}\tau_2)\frac{\tau_1}{\tau_2}\right]$$

$A_{21} = 10^8\ s^{-1}$ $\quad \tau_1 = 1\,ns,\ \tau_2 = 5\,ns$

$$\phi = 10^8 \cdot 5\times10^{-9}\left[1 + (1 - 10^8 \cdot 5\times10^{-9})\frac{1}{5}\right] = 0.55$$

$$g(\nu_0,\nu) = \frac{2/\pi\Delta\nu}{1 + \left[\dfrac{2(\nu - \nu_0)}{\Delta\nu}\right]^2}$$

At 1 FWHM from line center $g(\nu_0,\nu) = \dfrac{2}{5\pi\Delta\nu}$

For a naturally broadened system

$$\Delta\nu = \frac{A_1 + A_2}{2\pi} = \frac{10^9 + 0.2\times10^9}{2\pi} = 1.91\times10^8\ Hz$$

so $$I_s(\nu) = \frac{8\pi \times 6.626\times10^{-34} \times 2.997\times10^8 \times 5\pi \times 1.91\times10^8}{10^{-18} \times 0.55 \times 2}$$

$$= \underline{1.36\times10^4\ W/m^2}$$

(2.7) $$\tau = \frac{10^{-8}}{(1+P)}$$

gives a collision broadened linewidth

$$\Delta\nu = \frac{A}{2\pi} = \frac{1}{2\pi\tau} = \frac{1+P}{2\pi \times 10^{-8}}$$

$$\Delta\nu_D = 2\nu_0 \left(\frac{2kT \ln 2}{Mc^2}\right)^{1/2} = \frac{2}{\lambda}\left(\frac{2kT\ln 2}{M}\right)^{1/2}$$

$$M\ (\text{argon}) = 40 \times 1.67 \times 10^{-27}\ kg$$

Take $\lambda = 488$ nm

$$\Delta\nu_D = \frac{2}{488 \times 10^{-9}}\left(\frac{2 \times 1.38 \times 10^{-23} \times 200 \times \ln 2}{40 \times 1.67 \times 10^{-27}}\right)^{1/2}$$

$$= 3.102 \times 10^{9}\ Hz$$

Therefore collisional broadening = Doppler broadening when

$$\frac{1+P}{2\pi \times 10^{-8}} = 3.102 \times 10^{9} \Rightarrow P = 193.9 \text{ atmosphere}$$