(2.8) $\chi''(\nu)$ and $\chi'(\nu)$ vary as in figure (2.16)

The slowest phase velocity will occur where $\chi'(\nu)$ is a maximum, ie at $\nu = \nu_0 - \Delta\nu/2$

Since $\chi'(\nu) = \dfrac{2(\nu_0 - \nu)}{\Delta\nu}\chi''(\nu)$

$$\chi'_{max} = \chi''\left(\nu_0 - \frac{\Delta\nu}{2}\right)$$

$$\chi''\left(\nu_0 - \frac{\Delta\nu}{2}\right) = \frac{1}{2}\chi''(\nu_0)$$

Furthermore $\chi''(\nu_0) = -\dfrac{n^2\gamma(\nu_0)}{k}$

so $\chi'_{max} = \dfrac{n^2\gamma(\nu_0)}{2k}$

Take $n = 1$, $\gamma(\nu_0) = 1\ m^{-1}$ $k = 2\pi/\lambda_0$

$$\chi'_{max} = \frac{300\times10^{-9}}{4\pi} \Rightarrow \Delta k = \frac{k\chi'}{2n^2} = \frac{k}{2}\left(\frac{300\times10^{-9}}{4\pi}\right)$$

From Eq. (2.131) $c' = \dfrac{c}{1+\frac{\Delta k}{k}} = c\left(1-\dfrac{\Delta k}{k}\right)$

$$\Delta c = c - c' = \frac{c\Delta k}{k} = \frac{1}{2}\,\frac{2.998\times10^{8}\times300\times10^{-9}}{4\pi}$$

$\Delta c = 3.58\ m/s$

The wave slows up by 3.58 m/s

(2.9) $\Delta\nu = 10^9$ Hz

$N_2 = 5\times10^{16}$ m^{-3} $\qquad g_2 = g_1$

$N_1 = 5\times10^{15}$ m^{-3} $\qquad A_{21} = 10^8$

$\lambda_0 = 1\mu m$ $\qquad \tau_2 = 5\times10^{-9}$, $\tau_1 = 10^{-9}$

(i) $$\phi = A_{21}\tau_2\left[1 + (1 - A_{21}\tau_2)\frac{\tau_1}{\tau_2}\right]$$

$$= 10^8 \times 5\times10^{-9}\left[1 + (1 - 10^8 . 5\times10^{-9})\frac{1}{5}\right] = 0.55$$

$$I_s(\nu_0) = \frac{8\pi h\nu^3}{c^2\phi\, g(\nu_0,\nu)} = \frac{8\pi h\nu^3 . \pi\Delta\nu}{2c^2\phi}$$

$$= \frac{8\pi\times 6.626\times10^{-34}\times 2.997\times10^8\times\pi\times10^9}{2\times10^{-18}\times 0.55}$$

$$= 1.4254\times10^7 \text{ W m}^{-2}$$

(ii) $$\gamma(\nu) = (N_2 - N_1)\frac{c^2 A_{21}}{8\pi\nu^2}\, g(\nu_0,\nu)$$

$$= \frac{4.5\times10^{16}\times10^{-12}\times10^8}{8\pi}\, g(\nu_0,\nu)$$

$$= 1.79.10^{11} g(\nu_0,\nu) \quad m^{-1}$$

$$\chi'' = \frac{-n^2\gamma}{k}$$ Take $n = 1$

$$\chi'' = -\frac{1.79.10^{11} g(\nu_0,\nu)\,\lambda_0}{2\pi} = \frac{1.79\times10^5}{2\pi} g(\nu_0,\nu)$$

$$\chi' = \frac{2(\nu_0 - \nu)}{\Delta\nu}\chi''$$

(iii) At 1 FWHM from line centre

$$g(\nu_0, \nu) = \frac{2}{5\pi\Delta\nu}$$

Therefore $\chi'' = \frac{1.79 \times 10^5}{2\pi} \cdot \frac{2}{5\pi \cdot 10^9} = 3.63 \times 10^{-6}$

Therefore $\chi' = 2\chi'' = 7.255 \times 10^{-6}$

$$\Delta k = \frac{2\pi}{\lambda}\frac{\chi'}{2} = \frac{\pi \times 7.255 \times 10^{-6}}{10^{-6}} = 22.8\ m^{-1}$$

New phase velocity is $\frac{\omega}{(k+\Delta k)} = V_p = \frac{\omega}{k\left(1+\frac{\Delta k}{k}\right)} = \frac{\omega}{k}\left(1-\frac{\Delta k}{k}\right)$

Therefore Δ(phase velocity) is $-\frac{\omega}{k} \cdot \frac{\Delta k}{k} = -c \cdot \frac{\Delta k}{k}$

$$= -c \cdot \frac{22.8 \cdot 10^{-6}}{2\pi}$$

$$= \underline{4.456 \cdot 10^{-7}\ c}$$

(2.9) Continued

APPROXIMATE SOLUTION

(iv) If $\tau_1 = 1\,\mu s$ and $\tau_2 = 5\,ns$ the lower level will fill up by transitions from the upper and the population inversion will be transient.

Neglecting stimulated emission $\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2}$

$$\frac{dN_2}{dt} + \frac{N_2}{\tau_2} = R_2 \longrightarrow N_2 e^{t/\tau_2} = \int R_2 e^{t/\tau_2} dt$$

$$= R_2 \tau_2 e^{t/\tau_2} + \text{constant}$$

$N_2 = 0$ at $t = 0$. Therefore $N_2 = R_2 \tau_2 (1 - e^{-t/\tau_2})$

For small t $N_2 = R_2 t$

$$\frac{dN_1}{dt} = N_2 A_{21} - \frac{N_1}{\tau_1}$$

For small t $\frac{dN_1}{dt} = N_2 A_{21} = R_2 A_{21} t$

Therefore for small t $N_1 = \frac{1}{2} R_2 A_{21} t^2$

$$N_2 - N_1 = R_2 t - \frac{1}{2} R_2 A_{21} t^2 = R_2 t \left[1 - \frac{1}{2} A_{21} t \right]$$

inversion goes to zero when $t = \frac{2}{A_{21}} = 20\,ns$.

EXACT SOLUTION

(iv) $R_2 \rightarrow$ ———— N_2 Assume that at time $t=0$, $N_2 = N_1 = 0$

———— N_1

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} \Rightarrow \frac{dN_2}{dt} + \frac{N_2}{\tau_2} = R_2$$

Multiply each term by e^{t/τ_2}

$$\frac{dN_2}{dt} e^{t/\tau_2} + \frac{N_2}{\tau_2} e^{t/\tau_2} = R_2 e^{t/\tau_2}$$

Both sides can now be integrated

$$N_2 e^{t/\tau_2} = R_2 \tau_2 e^{t/\tau_2} + \text{constant}$$

From $N_2 = 0$ at $t=0$ $\Rightarrow$ constant $= -R_2\tau_2$

Therefore $N_2 = R_2 \tau_2 (1 - e^{-t/\tau_2})$

$$\frac{dN_1}{dt} = N_2 A_{21} - \frac{N_1}{\tau_1} \Rightarrow \frac{dN_1}{dt} + \frac{N_1}{\tau_1} = R_2 A_{21} \tau_2 (1 - e^{-t/\tau_2})$$

gives $N_1 e^{t/\tau_1} = \int R_2 A_{21} \tau_2 e^{t/\tau_1} (1 - e^{-t/\tau_2})\, dt$

$$= R_2 A_{21} \tau_2 \tau_1 e^{t/\tau_1} - \frac{R_2 A_{21} \tau_2 e^{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)t}}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} + \text{const}$$

$$N_1 = R_2 A_{21} \tau_2 \tau_1 - \frac{R_2 A_{21} \tau_2 e^{-t/\tau_2}}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} + \text{const} \cdot e^{-t/\tau_1}$$

From $N_1 = 0$ at $t=0$ const. $= \frac{R_2 A_{21} \tau_2}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} - R_2 A_{21} \tau_2 \tau_1$

Therefore

$$N_1 = R_2 A_{21} \tau_2 \tau_1 \left(1 - e^{-t/\tau_1}\right) + \frac{R_2 A_{21} \tau_2}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} \left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$$

The population inversion is $N_2 - N_1$

$$N_2 - N_1 = R_2 \tau_2 \left(1 - e^{-t/\tau_2}\right) - R_2 A_{21} \tau_2 \tau_1 \left(1 - e^{-t/\tau_1}\right) - \frac{R_2 A_{21} \tau_2}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} \left(e^{-t/\tau_1} - e^{-t/\tau_1}\right)$$

If this inversion is positive then the system will oscillate with appropriate feedback. However, for longer times, since $\tau_1 > \tau_2$ the inversion will disappear – this is a so-called "SELF-TERMINATING LASER"

For short times

$$N_2 - N_1 = R_2 t - R_2 A_{21} \tau_2 t - \frac{R_2 A_{21} \tau_2}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} \left(\frac{t}{\tau_2} - \frac{t}{\tau_1}\right)$$

$$= R_2 t, \quad \text{which is} > 0$$

For longer times, such that $e^{-t/\tau_1} \to 0$

$$N_2 - N_1 = R_2 \tau_2 \left(1 - e^{-t/\tau_2}\right) + \frac{R_2 A_{21} \tau_2}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} e^{-t/\tau_2}$$

At very long times

$$N_2 - N_1 = R_2 \tau_2 - R_2 A_{21} \tau_2 \tau_1 - \frac{R_2 A_{21} \tau_2}{\frac{1}{\tau_1} - \frac{1}{\tau_2}}$$

As shown on the accompanying Mathcad plot the inversion disappears in this case at f ~ 19 ns

(2.9) (iv)

$R2 := 10^{20}$

$\tau 2 := 5 \cdot 10^{-9}$

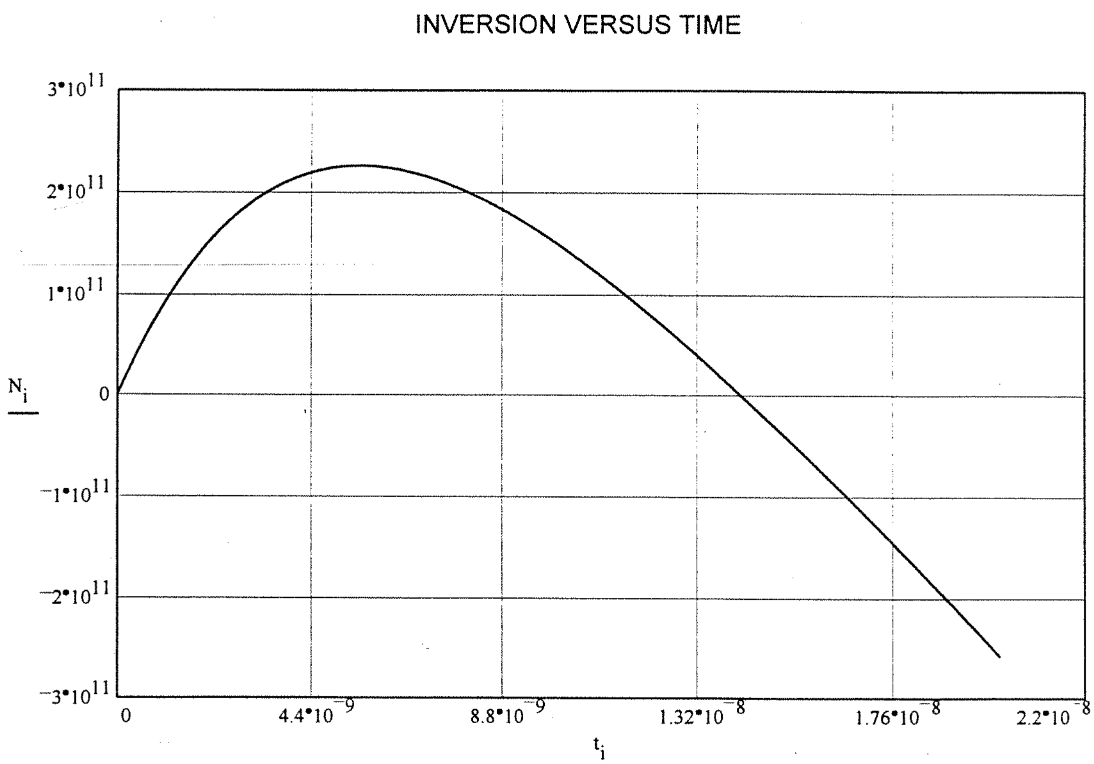
$\tau 1 := 10^{-6}$

$A21 := 10^{8}$

$i := 1, 2 .. 1000$

$t_i := (2 \cdot i) \cdot \frac{10^{-8}}{999}$

$$N_i := R2 \cdot \tau 2 \cdot \left(1 - e^{-\frac{t_i}{\tau 2}}\right) - R2 \cdot A21 \cdot \tau 1 \cdot \tau 2 \cdot \left(1 - e^{-\frac{t_i}{\tau 1}}\right) - R2 \cdot A21 \cdot \tau 2 \cdot \frac{e^{-\frac{t_i}{\tau 1}} - e^{-\frac{t_i}{\tau 2}}}{\left(\frac{1}{\tau 1} - \frac{1}{\tau 2}\right)}$$



(2.10) Use equation (2.128)

$$k' = \omega\sqrt{\mu_0 \varepsilon_r \varepsilon_0}\sqrt{1+\frac{\chi(\omega)}{\varepsilon_r}} = k\sqrt{1+\frac{\chi'(\omega)}{\varepsilon_r} - i\frac{\chi''(\omega)}{\varepsilon_r}}$$

if $|\chi'(\omega)|$ is not $<<$ ε_r we cannot expand the square root as was done in Eq (2.129)

Write $\quad k' = k\sqrt{a - ib} \qquad (1)$

where $\quad a = 1 + \frac{\chi'(\omega)}{\varepsilon_r} \qquad b = \frac{\chi''(\omega)}{\varepsilon_r}$

$$k' = \sqrt{k^2 a - i k^2 b} \equiv \sqrt{\rho\, e^{i\phi}}$$

where $\quad \rho = k^2 (a^2+b^2)^{1/2}$

$\quad \phi = \arctan\,(b/a)$

$$k' = \sqrt{\rho}\; e^{i\phi/2} = k\,(a^2+b^2)^{1/4} e^{i\phi/2}$$

$$k' = k\,(a^2+b^2)^{1/4}\cos\phi/2 + i\,k(a^2+b^2)^{1/4}\sin\phi/2$$

In this case $\quad c' = \frac{\omega}{k + \Delta k}$

where $\quad \Delta k = k(a^2+b^2)^{1/4}\cos(\phi/2) - k$

$$\gamma = -2k\,(a^2+b^2)^{1/4}\sin\phi/2$$

where $\quad (a^2+b^2) = \left[1+\frac{\chi'(\omega)}{\varepsilon_r}\right]^2 + \left[\frac{\chi''(\omega)}{\varepsilon_r}\right]^2$

$\quad \tan\phi = \chi'(\omega)/[\varepsilon_r + \chi'(\omega)]$