ENEE 496 Spring 2004

Problem Set 1. Due February 17, 2004

(1) A particle has a lineshape function for frequencies $|\nu - \nu_0| \leq \pi\Delta/2$

$$g(\nu_0, \nu) = C\cos(\frac{\nu_0 - \nu}{\Delta}).$$

(a) What is the value of C needed to normalize this function?
(b) What is the fullwidth at half maxiumum (FWHM) of this function? This is the frequency spacing between the two frequencies where $g(\nu_0, \nu) = \frac{1}{2}g(\nu_0, \nu_0)$.

(2) For the line shape function in (1) if A_{21} is 10^8 s^{-1}, what is the stimulated emission rate produced by a monchromatic input signal at a wavelength λ_0=1μm of 1W/m^2 at a frequency 1 FWHM from the line center?

(3) A group of particles with N_2=10^{18} m^{-3} and N_1=10^{17} m^{-3}, which extends from z=0 to ∞ is irradiated beginning at $t = 0$ with a monochromatic plane wave at line center of intensity 1W/m^2. The lineshape and A_{21} are the same as in questions (1) and (2).
(a) Plot the population difference $N_2 - N_1$ as a function of time at z=0.
(b) At what time does $N_2 - N_1$=0?

(4) Plot the shape of the black body radiation distribution $\rho(\nu)$ for temperatures of 1000K, 2000K, up to 10,000K. From these graphs determine the value of the constant F in the relation $\nu_{max}T = F$.

(5) Davis Problem 1.1

(6) Davis Problem 1.3

(7) Davis Problem 1.4

(8) Davis Problem 1.7

(9) Davis Problem 1.9

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(1)

$$g(\nu_0, \nu) = C\cdot\cos\left(\frac{\nu_0 - \nu}{\Delta}\right)$$

$$\int_{\nu_0-\pi\cdot\frac{\Delta}{2}}^{\nu_0+\pi\cdot\frac{\Delta}{2}} C\cdot\cos\left(\frac{\nu_0-\nu}{\Delta}\right) d\nu = 2\cdot\Delta\cdot C$$

Therefore for normalization C=1/2Δ

To find FWHM

$$\frac{\nu_0 - \nu}{\Delta} = \mathrm{acos}\left(\frac{1}{2}\right)$$

gives

$$\nu = \nu_0 - \frac{1}{3}\cdot\pi\cdot\Delta$$

$$\mathrm{FWHM} = \frac{2}{3}\cdot\pi\cdot\Delta$$

(2) Stimulated emission rate is $W_{21} = B_{21}\cdot\frac{I}{c}\cdot g\left(\nu_0, \nu_0 + \frac{2}{3}\cdot\pi\,\Delta\right)$

$A_{21} := 10^8$ $\quad c := 2.998\cdot 10^8$ $\quad h := 6.626\cdot 10^{-34}$ $\quad \lambda := 10^{-6}$

$\nu := \frac{c}{\lambda}$ $\quad I := 1$

At 1 FWHM from line center $g(\nu_0,\nu)$=0 $\quad B_{21} := \frac{c^3\cdot A_{21}}{8\cdot\pi\cdot h\cdot\nu^3}$

$W_{21} = 0$

(3) At line center $g(\nu_0,\nu_0) = \frac{1}{2\cdot\Delta}$

Set FWHM=$A_{21}/2\pi$ $\Delta := \frac{3\cdot A_{21}}{4\cdot\pi^2}$ $\Delta = 7.599 \times 10^6$

Write $g := \frac{1}{2\cdot\Delta}$ $B_{12} := B_{21}$

$g = 6.58 \times 10^{-8}$

$$\frac{d}{dt}N_2 = -A_{21}\cdot N_2 - B_{21}\cdot\frac{I}{c}\cdot g\cdot N_2 + B_{12}\cdot\frac{I}{c}\cdot g\cdot N_1$$

$$\frac{d}{dt}N_1 = A_{21}\cdot N_2 + B_{21}\cdot\frac{I}{c}\cdot g\cdot N_2 - B12\cdot\frac{I}{c}\cdot g\cdot N_1$$

Solve these two equations by 4th order Runga-Kutta

$NN := \begin{pmatrix} 10^{18} \\ 0 \end{pmatrix}$ Population matrix

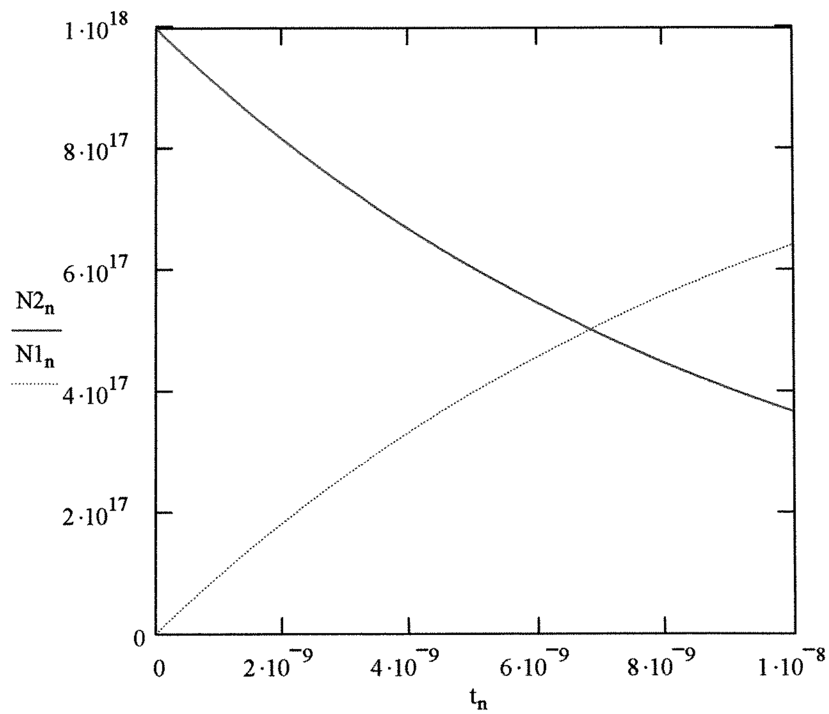
$$D(t,NN) := \begin{bmatrix} -A_{21}\cdot NN_0 - B_{21}\cdot\frac{I}{c}\cdot g\cdot NN_0 + B_{12}\cdot\frac{I}{c}\cdot g\cdot NN_1 \\ \left(A_{21}\cdot NN_0 + B_{21}\cdot\frac{I}{c}\cdot g\cdot NN_0 - -B_{12}\cdot\frac{I}{c}\cdot g\cdot NN_1\right) \end{bmatrix}$$

$Z := \text{rkfixed}\left(NN, 0, 1\cdot 10^{-8}, 1000, D\right)$

$n := 0..1000$

$N2_n := Z_{n,1}$ $N1_n := Z_{n,2}$ $t_n := Z_{n,0}$

N_2 and N_1 v. time



$N2 = N1$ **when t=6.85ns**

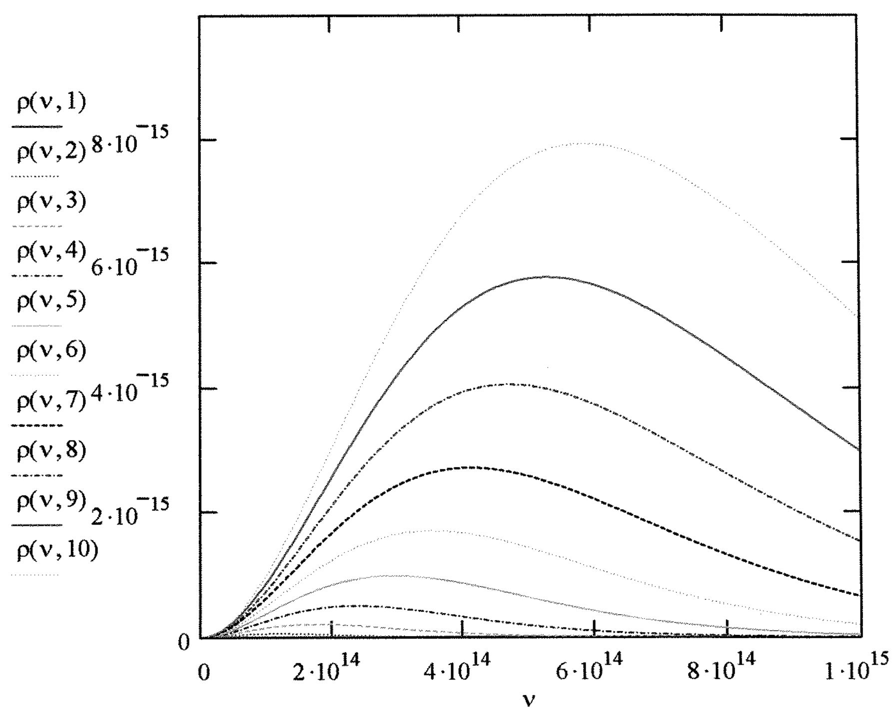
(4) $\nu := 10^{12}, 2\cdot10^{12} .. 10^{15}$ $\quad k := 1.38\cdot10^{-23}$

$i := 1,2 .. 10$

$T_i := 1000\cdot i$

$$\rho(\nu,i) := \frac{8\cdot\pi\cdot h\cdot\nu^3}{c^3\left(e^{h\cdot\frac{\nu}{k\cdot T_i}} - 1\right)}$$

Curves for T=1000,2000, 3000, ...10,000K



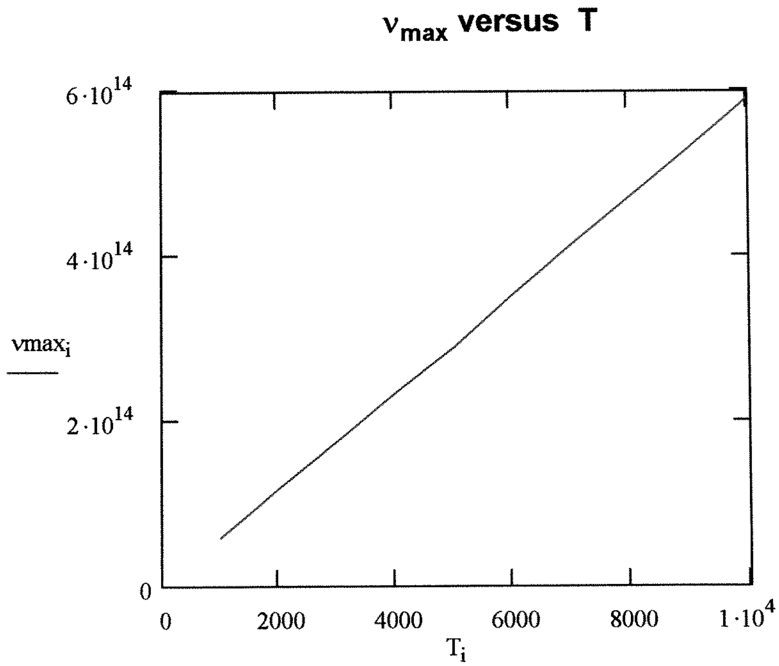
Fron curves

$vmax_{10} := 5.93\cdot10^{14}$ $\quad vmax_9 := 5.29\cdot10^{14}$ $\quad vmax_8 := 4.71\cdot10^{14}$

$vmax_7 := 4.13\cdot10^{14}$ $\quad vmax_6 := 3.53\cdot10^{14}$ $\quad vmax_5 := 2.89\cdot10^{14}$

$vmax_4 := 2.34\cdot10^{14}$ $\quad vmax_3 := 1.76\cdot10^{14}$ $\quad vmax_2 := 1.19\cdot10^{14}$

$vmax_1 := 0.59\cdot10^{14}$



Slope of line is approximately $5.867*10^{10}$

So $$\frac{vmax}{T} = 5.867 \cdot 10^{10}$$

(5)

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(1.1) Group refractive index is $n_g = n - \lambda \frac{dn}{d\lambda}$

(i) $\frac{dn}{d\lambda} = \frac{dn}{d\nu}\frac{d\nu}{d\lambda}$ $\quad \nu = \frac{c}{\lambda} \Rightarrow \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$

Therefore $n_g = n - \lambda \frac{dn}{d\nu}\left(-\frac{c}{\lambda^2}\right) = n + \nu \frac{dn}{d\nu}$ Q.E.D

(ii) The total number of modes with frequency less than ν is

$$N_\nu = \frac{8\pi\nu^3}{3c^3} = \frac{8\pi\nu^3 n^3}{3c_0^3}$$

The mode density is $\rho_\nu = \frac{dN_\nu}{d\nu}$

Therefore $\rho_\nu = \frac{8\pi\nu^2 n^3}{c_0^3} + \frac{8\pi\nu^3 n^2}{c_0^3}\frac{dn}{d\nu}$

$$\rho_\nu = \frac{8\pi\nu^2 n^2}{c_0^3}\left(n + \nu\frac{dn}{d\nu}\right) = \frac{8\pi\nu^2 n^2 n_g}{c_0^3} \quad \text{Q.E.D}$$

(iii) The ratio $\frac{A_{21}}{B_{21}}$ corresponds to a stimulated emission rate determined by an energy density of one photon per mode

$$\frac{A_{21}}{B_{21}} = \frac{8\pi\nu^2 n^2 n_g}{c_0^3}\cdot h\nu = \frac{8\pi h\nu^3 n^2 n_g}{c_0^3} \quad \text{Q.E.D}$$

(1.2) Photon flux is $N = \frac{I}{h\nu} = \frac{I\lambda}{hc}$

$h = 6.626 \cdot 10^{-34}$ Js

(a) At 100 nm $N = \frac{100 \times 100 \times 10^{-9}}{6.626 \cdot 10^{-34} \times 2.998 \cdot 10^{8}}$

$N = 5.034 \times 10^{19}$ photons $m^{-2} s^{-1}$

(b) At 100 μm $N = \frac{100 \times 100 \times 10^{-6}}{6.626 \times 10^{-34} \times 2.998 \times 10^{8}}$

$N = 5.034 \times 10^{22}$ photons $m^{-2} s^{-1}$

(6)

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(1.3) The mode density ($m^{-3} Hz^{-1}$) is

$$p(\nu) = \frac{8\pi\nu^2}{c^3}$$

The number of modes between ν_1 and ν_2 is

$$N_\nu = \int_{\nu_1}^{\nu_2} \frac{8\pi\nu^2}{c^3} d\nu = \frac{8\pi}{3c^3}\left(\nu_2^3 - \nu_1^3\right) = \frac{8\pi}{3}\left(\frac{1}{\lambda_2^3} - \frac{1}{\lambda_1^3}\right)$$

The visible region extends from $\lambda_2 \simeq 400$ nm to $\lambda_1 \simeq 700$ nm

Therefore, the total number of modes is

$$N_\nu = \frac{8\pi}{3}\left(\frac{10^{27}}{400^3} - \frac{10^{27}}{700^3}\right) = 1.065 \cdot 10^{20} \text{ modes m}$$

(7)

DAVIS (1.4) $e(\nu)$ for blackbody radiation is

$$e(\nu) = \frac{8\pi h\nu^3}{c^3}\left(\frac{1}{e^{h\nu/kT}-1}\right)$$

Convert $e(\nu)$ to $e(\lambda)$ so that $e(\nu)d\nu = e(\lambda)d\lambda$

Remember that $\lambda = \frac{c}{\nu}$ so $d\lambda = (-)\frac{c}{\nu^2}d\nu$

Therefore $e(\lambda) = \frac{8\pi hc}{\lambda^5}\left(\frac{1}{e^{hc/\lambda kT}-1}\right)$

$$\frac{d}{d\lambda}[e(\lambda)] = 8\pi hc\left[\frac{-5}{\lambda^6(e^{hc/\lambda kT}-1)} + \frac{\frac{hc}{\lambda^2 kT}e^{hc/\lambda kT}}{(e^{hc/\lambda kT}-1)^2\lambda^5}\right]$$

To find maximum of $e(\lambda)$ $\frac{d}{d\lambda}[e(\lambda)] = 0$

This requires $-5(e^{hc/\lambda kT}-1) + \frac{hc}{\lambda kT}e^{hc/\lambda kT} = 0$

Put $hc/\lambda kT = x \Rightarrow \left(1-\frac{x}{5}\right)e^x = 1$

Solution is $x = 4.9651$ see Mathcad solution

Therefore $\frac{hc}{\lambda_{max}kT} = 4.9651$

$$\lambda_{max}T = \frac{hc}{4.9651k} = \frac{6.6261\times10^{-34}\times2.998\times10^8}{4.9651\times1.3807\times10^{-23}}$$

$$= 2.898\cdot10^{-3}\ \text{m K}$$

$$= 2898\ \mu\text{m K}$$

This answer is not obtained if ν_{max} is calculated

In fact $\nu_{max} = \frac{2.8kT}{h}$

1.4 define variables

$k := 1.380658 \cdot 10^{-23}$ Boltzmann's constant

$c := 2.99792458 \cdot 10^{8}$ velocity of light

$h := 6.6260755 \cdot 10^{-34}$

$$x = h \cdot \frac{c}{k \cdot \lambda mT}$$

$x := 5$

Given

$$5 \cdot e^{x} - 5 - x \cdot e^{x} = 0$$

$a := \mathrm{Find}(x)$

$a = 4.9651142$

$$\lambda mT := h \cdot \frac{c}{k \cdot a}$$

$\lambda mT = 0.0028978$ 2898 micrometer K Q.E.D.

$$\rho_{\lambda} = \frac{8 \cdot \pi \cdot h \cdot c}{\lambda^{5} \cdot \left(e^{h \cdot \frac{c}{\lambda \cdot k \cdot T}} - 1 \right)}$$

(8).

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(1.7) $\rho(\lambda) = \dfrac{8\pi hc}{\lambda^5(e^{hc/\lambda kT}-1)}$

Total energy is $\int_{\lambda_1}^{\lambda_2} \rho(\lambda)\,d\lambda = u$

In this case can be approximated as

$u = \rho(\lambda_0)\,\Delta\lambda$ where $\lambda_0 = 10.6\,\mu m$; $\Delta\lambda = 0.2\,\mu m$

1.7 define variables

$k := 1.380658 \cdot 10^{-23}$ **Boltzmann's constant**

$c := 2.99792458 \cdot 10^{8}$ **velocity of light**

$h := 6.6260755 \cdot 10^{-34}$

$T := 3000$

$\lambda 0 := 10.6 \cdot 10^{-6}$

$\Delta\lambda := 0.2 \cdot 10^{-6}$

$$\rho := 8 \cdot \pi \cdot h \cdot \frac{c}{\lambda 0^5 \cdot \left(e^{h \cdot \frac{c}{\lambda 0 \cdot k \cdot T}} - 1\right)}$$

$E := \rho \cdot \Delta\lambda$

$E = 0.000013041$ **13.051 microJoule per cubic m**

Exact calculation by integration

$$\rho(\lambda) := 8 \cdot \pi \cdot h \cdot \frac{c}{\lambda^5 \cdot \left(e^{h \cdot \frac{c}{\lambda \cdot k \cdot T}} - 1\right)}$$

$\lambda 1 := 10.5 \cdot 10^{-6}$

$\lambda 2 := 10.7 \cdot 10^{-6}$

$$E := \int_{\lambda 1}^{\lambda 2} \rho(\lambda)\,d\lambda$$

$E = 0.000013044$ **13.044 microJoule per cubic m**

(9)
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(1.9) The stimulated emission rate is

$$W_{21} = B_{21}\,\rho(\nu) = \frac{c^3 A_{21}}{8\pi h\nu^3}\,\rho(\nu)$$

The spontaneous emission rate is A_{21}

For these to be equal

$$\frac{8\pi h\nu^3}{c^3\left(e^{h\nu/kT}-1\right)} = \frac{8\pi h\nu^3}{c^3}$$

which implies $e^{h\nu/kT} - 1 = 1$

and $\frac{h\nu}{kT} = \ln 2$

$$T = \frac{h\nu}{k\ln 2} = \frac{6.626\times10^{-34}\times 2.998\times10^{8}}{1.38\times10^{-23}\times \ln 2\times 10^{-6}} = \underline{20{,}767\ K}$$