ENEE 496 Spring 2004

Problem Set 2. Due March 4, 2004

(1) Davis Problem 2.1

(2) Davis Problem 2.3

(3) Davis Problem 2.4

(4) Davis Problem 2.5

(5) Davis Problem 2.6

(6) Davis Problem 2.7

(7) Davis Problem 2.8

(8) Davis problem 2.9

(2.1)

$\gamma_0(\nu_0) = 0.6\ m^{-1}$ $\quad l = 3m$

$I_s(\nu_0) = 3\ W\ m^{-2}$

$I_0 = 2.5\ W$

(a) Input at line center, no saturation included

$I = I_0 e^{\gamma_0 l} = 2.5 e^{1.8} = \underline{15.124\ W\ m^{-2}}$

(b) Input ~ 1 FWHM from line center

$$\gamma(\nu) = \frac{\gamma_0(\nu_0)}{1 + \left[\frac{2(\nu - \nu_0)}{\Delta\nu}\right]^2} = \frac{\gamma_0(\nu_0)}{5}$$

so $\gamma(\nu) = 0.12\ m^{-1}$

$I = I_0 e^{0.36} = 2.5 e^{0.36} = \underline{3.5833\ W\ m^{-2}}$

(c)

$$I = I_0 e^{\gamma_0(\nu_0) l - (I - I_0)/I_s} \qquad \text{(A)}$$

I lies somewhere between 2.5 & 15.124 $W\ m^{-2}$

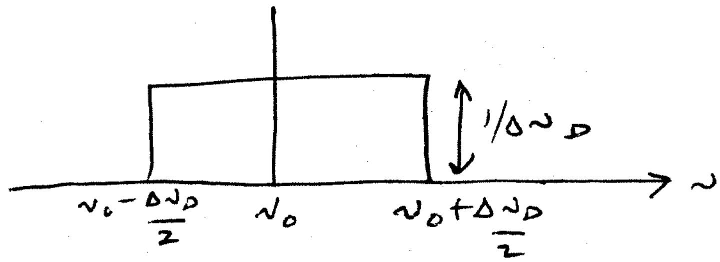
Mathcad gives $I = \underline{5.522\ W\ m^{-2}}$

(d) 1 FWHM from line center $\gamma_0 = 0.12\ m^{-1}$

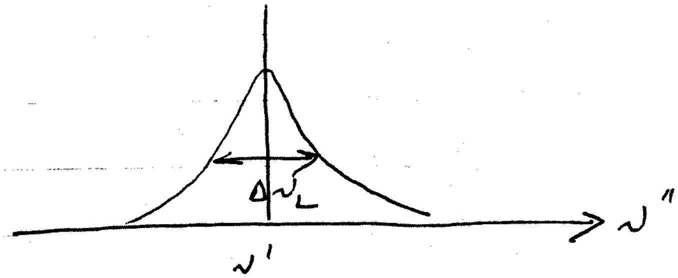
$I_s(\nu) \propto \frac{1}{g(\nu_0, \nu)}$ so $I_s(\nu) = 15\ W\ m^{-2}$

Solve Eq (A) again $\qquad I = \underline{3.379\ W\ m^{-2}}$

(2.3) $g_D(\nu_0, \nu)$



$g_L(\nu', \nu'')$



overall lineshape is

$$g_T(\nu_0, \nu) = \frac{1}{\Delta\nu_D} \int_{\nu_0 - \frac{\Delta\nu_D}{2}}^{\nu_0 + \frac{\Delta\nu_D}{2}} \frac{2}{\pi \Delta\nu_L} \frac{d\nu''}{1 + \left[\frac{2(\nu - \nu'')}{\Delta\nu_L}\right]^2}$$

Put $\frac{2(\nu - \nu'')}{\Delta\nu_L} = x \qquad d\nu'' = -\frac{\Delta\nu_L}{2} dx$

$$g_T(\nu_0, \nu) = \frac{-1}{\pi \Delta\nu_D} \int_{\frac{2}{\Delta\nu_L}\left(\nu - \nu_0 + \frac{\Delta\nu_D}{2}\right)}^{\frac{2}{\Delta\nu_L}\left(\nu - \nu_0 - \frac{\Delta\nu_D}{2}\right)} \frac{dx}{1 + x^2}$$

$$= \frac{1}{\pi \Delta\nu_D} \arctan x \Big|_{\frac{2}{\Delta\nu_L}\left(\nu - \nu_0 - \frac{\Delta\nu_D}{2}\right)}^{\frac{2}{\Delta\nu_L}\left(\nu - \nu_0 + \frac{\Delta\nu_D}{2}\right)}$$

$$g_T(\nu_0, \nu) = \frac{1}{\pi \Delta\nu_D}\left[\arctan \frac{2}{\Delta\nu_L}\left(\nu - \nu_0 + \frac{\Delta\nu_D}{2}\right) - \arctan \frac{2}{\Delta\nu_L}\left(\nu - \nu_0 - \frac{\Delta\nu_D}{2}\right)\right]$$

See following Mathcad plots

$i := 1, 2 .. 1001$

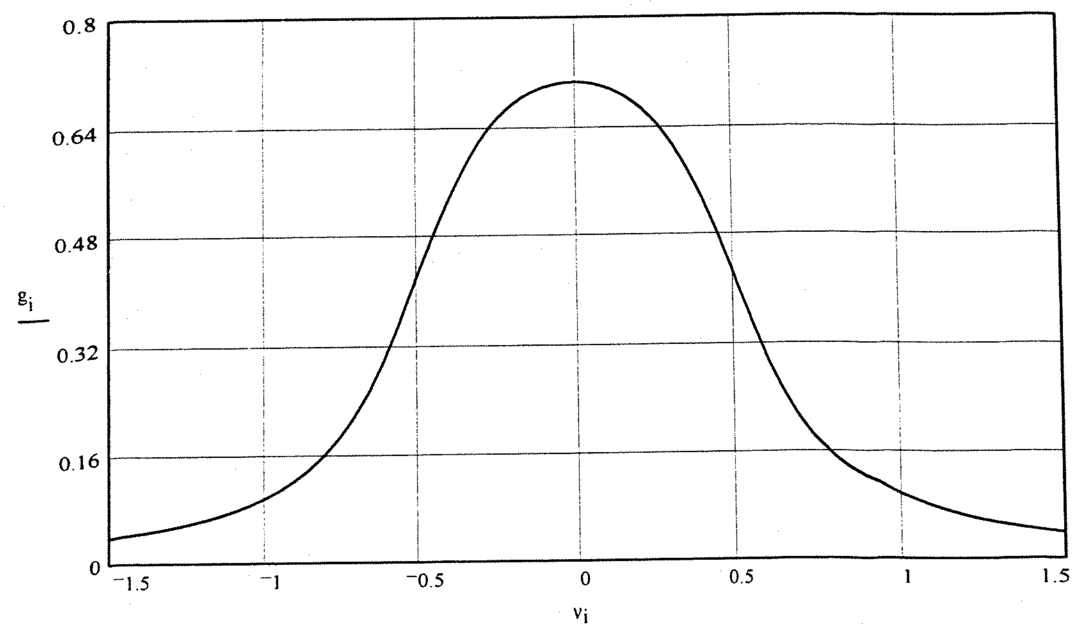
$\Delta\nu D := 1$

$\Delta\nu L := 0.5$

$$\nu_i := -1.5 \cdot \Delta\nu D + \frac{i-1}{1000} \cdot 3 \cdot \Delta\nu D$$

$$g_i := \frac{1}{\pi \cdot \Delta\nu D} \cdot \left[\mathrm{atan}\left[\frac{2}{\Delta\nu L} \cdot \left(\frac{\Delta\nu D}{2} + \nu_i \right) \right] - \mathrm{atan}\left[\frac{2}{\Delta\nu L} \cdot \left(\nu_i - \frac{\Delta\nu D}{2} \right) \right] \right]$$

$\Delta\nu_D/\Delta\nu_L=2$



$\longrightarrow \frac{\nu - \nu_0}{\Delta\nu_D}$

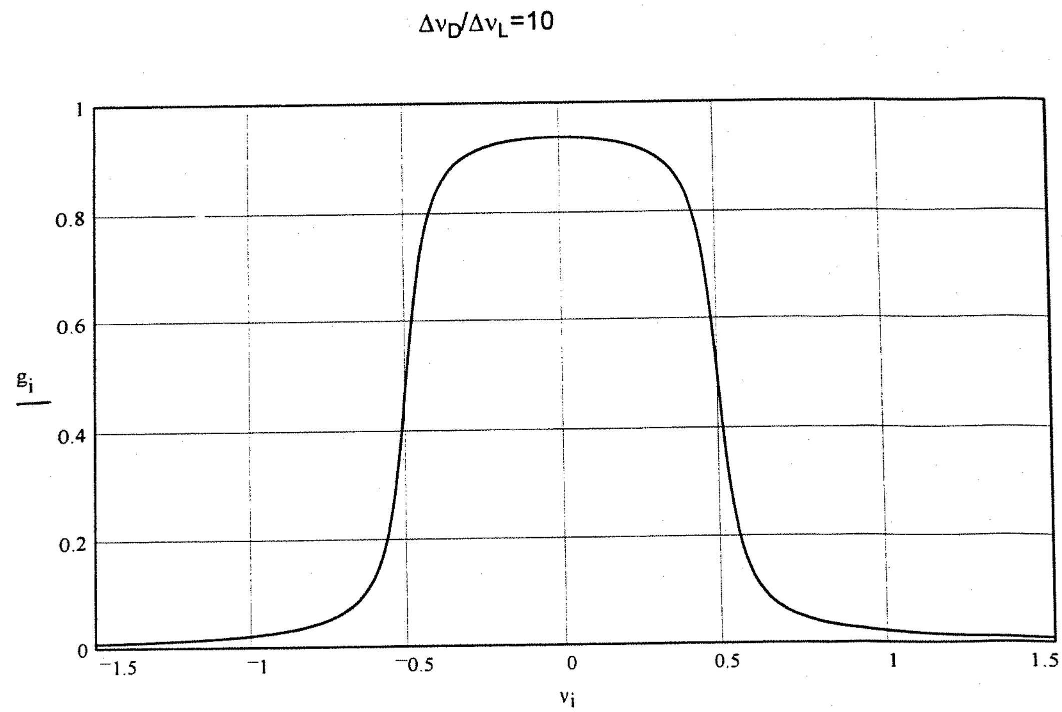
$i := 1, 2 .. 1001$

$\Delta\nu D := 1$

$\Delta\nu L := 0.1$

$$\nu_i := -1.5 \cdot \Delta\nu D + \frac{i-1}{1000} \cdot 3 \cdot \Delta\nu D$$

$$g_i := \frac{1}{\pi \cdot \Delta\nu D} \cdot \left[\operatorname{atan}\left[\frac{2}{\Delta\nu L} \cdot \left(\frac{\Delta\nu D}{2} + \nu_i \right) \right] - \operatorname{atan}\left[\frac{2}{\Delta\nu L} \cdot \left(\nu_i - \frac{\Delta\nu D}{2} \right) \right] \right]$$



$\longrightarrow \dfrac{\nu - \nu_0}{\Delta\nu_D}$

$i := 1, 2 .. 1001$

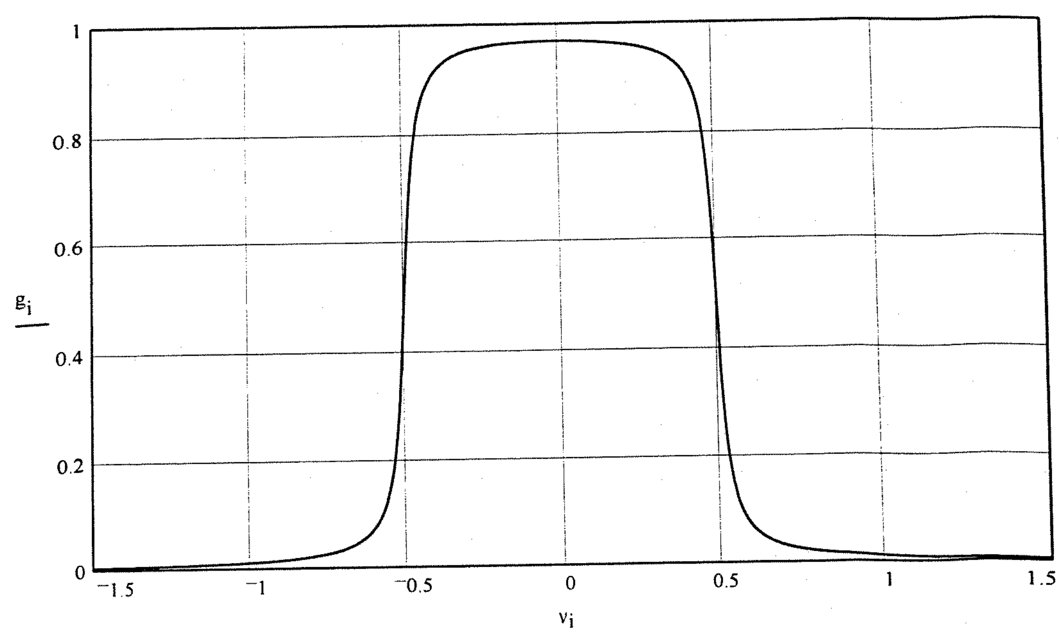
$\Delta\nu D := 1$

$\Delta\nu L := 0.05$

$$\nu_i := -1.5\cdot\Delta\nu D + \frac{i-1}{1000}\cdot 3\cdot\Delta\nu D$$

$$g_i := \frac{1}{\pi\cdot\Delta\nu D}\cdot\left[\operatorname{atan}\left[\frac{2}{\Delta\nu L}\cdot\left(\frac{\Delta\nu D}{2}+\nu_i\right)\right]-\operatorname{atan}\left[\frac{2}{\Delta\nu L}\cdot\left(\nu_i-\frac{\Delta\nu D}{2}\right)\right]\right]$$

$\Delta\nu_D/\Delta\nu_L=20$



$\longrightarrow \dfrac{\nu - \nu_0}{\Delta\nu_D}$

(2.4) The stimulated emission rate is

$$W_{21}(\omega) = \frac{c^3 A_{21}}{8\pi h \omega^3} g(\omega_0, \omega)\, \rho(\omega) \qquad sec^{-1}\ Hz^{-1}$$

For "white" input radiation

$$W_{21} = \int W_{21}(\omega)\, d\omega = \frac{c^3 A_{21}}{8\pi h \omega^3} \rho(\omega) \int_{-\infty}^{\infty} g(\omega_0, \omega)\, d\omega$$

$$= \frac{c^3 A_{21}}{8\pi h \omega^3} \rho(\omega)$$

$$= \frac{\lambda^3 A_{21}}{8\pi h} \rho(\omega)$$

For example given $W_{21} = \dfrac{(3\times10^8)^3\ 10^8 \times 1}{8\pi \times 6.6\times10^{-34} \times 10^{+2}}$

$$= \underline{1.6277\times10^{23}\, s^{-1}}$$

For monochromatic input

$$W_{21} = \int W_{21}(\omega)\, d\omega = \frac{c^3 A_{21}}{8\pi h \omega^3} g(\omega_0, \omega)\, \rho_\omega$$

$\rho = \dfrac{I}{c}$ $\quad \therefore W_{21} = \dfrac{c^2 A_{21}}{8\pi h \omega^3} I\, g(\omega_0, \omega)$

In the example given $\quad g(\omega_0, \omega) = \dfrac{(2/\pi\Delta\omega)}{5}$

$$\therefore W = \frac{c^2 A_{21} I}{8\pi h \omega^3}\left(\frac{2}{5\pi\Delta\omega}\right) = \frac{(3\times10^8)^2 \times 10^8 \times 1}{20\pi^2 \times 6.6\times10^{-34} \times 10^{+2} \times 10^9}$$

$$= \underline{6.9\times10^4\ s^{-1}}$$

(2.5) $\Delta\nu = 10^{9}$ Hz

$\gamma_0(\nu_0) = 1\, m^{-1}$

$I_s(\nu_0) = 1\, W\, m^{-2}$

$\ell = 0.5$ m

$I_0 = 2\, W\, m^{-2}$

(a) 500 MHz from line center $\gamma_0 = 0.5\, m^{-1}$, $I_s = 2\, W\, m^{-2}$

w/o saturation $I_{out} = 2e^{0.25} = 2.568\, W\, m^{-2}$

(b) With saturation by Mathcad (see problem 2.1)

$I_{out} = 2.258\, W\, m^{-2}$

(c) At line center $\gamma_0 = 1\, m^{-1}$, $I_s = 1\, W\, m^{-2}$

with saturation by mathcad (see problem 2.1)

$I_{out} = 2.342\, W\, m^{-2}$

(d) The 2 signals above and below line center see the same saturation intensity

Intensity of radiation above line center is I_1

Intensity of radiation below line center is I_2

$$\frac{1}{I_1}\frac{dI_1}{dz} = \frac{\gamma_0}{1+(I_1+I_2)I_s} \quad \text{(i)}$$

Write $I = I_1 + I_2$

$$\frac{1}{I_2}\frac{dI_2}{dz} = \frac{\gamma_0}{1+(I_1+I_2)I_s} \quad \text{(ii)}$$

gives $$\frac{1}{I}\frac{dI}{dz} = \frac{\gamma_0}{(1+I/I_s)}$$

Substitute in (i)

$$\frac{1}{I_1}\frac{dI_1}{dz} = \frac{\gamma_0}{1+(I_1^\circ+I_2^\circ)e^{\gamma_0 l-(I_1+I_2-I_1^\circ-I_2^\circ)/I_s}}$$

Substitute in (ii)

$$\frac{1}{I_2}\frac{dI_2}{dz} = \frac{\gamma_0}{1+(I_1^\circ+I_2^\circ)e^{\gamma_0 l-(I_1+I_2-I_1^\circ-I_2^\circ)/I_s}}$$

where I_1° & I_2° are the input intensities

This is the exact solution and would need to be solved numerically with $I_1 = 2\ Wm^{-2}$, $I_2 = 1\ Wm^{-2}$
$I_s = 2\ W m^{-2}$

(2.6) Saturation Intensity is $I_s(\nu) = \dfrac{8\pi h\nu^3}{c^2 \phi\, g(\nu_0,\nu)}$

$$\phi = A_{21}\tau_2\left[1 + (1 - A_{21}\tau_2)\frac{\tau_1}{\tau_2}\right]$$

$A_{21} = 10^8\ s^{-1}$ $\quad \tau_1 = 1\,ns, \quad \tau_2 = 5\,ns$

$$\phi = 10^8 \cdot 5\times10^{-9}\left[1 + (1 - 10^8 \cdot 5\times10^{-9})\frac{1}{5}\right] \qquad = 0.55$$

$$g(\nu_0,\nu) = \frac{2/\pi\Delta\nu}{1 + \left[\dfrac{2(\nu-\nu_0)}{\Delta\nu}\right]^2}$$

At 1 FWHM from line center $\quad g(\nu_0,\nu) = \dfrac{2}{5\pi\Delta\nu}$

For a naturally broadened system

$$\Delta\nu = \frac{A_1 + A_2}{2\pi} = \frac{10^9 + 0.2\times10^9}{2\pi} = 1.91\times10^8\ Hz$$

$$\text{so} \quad I_s(\nu) = \frac{8\pi \times 6.626\times10^{-34} \times 2.997\times10^8 \times 5\pi \times 1.91\times10^8}{10^{-18}\times0.55\times2}$$

$$= \underline{1.36\times10^4\ W/m^2}$$

(2.7) $\tau = \dfrac{10^{-8}}{(1+P)}$

gives a collision broadened linewidth

$$\Delta\nu = \frac{A}{2\pi} = \frac{1}{2\pi\tau} = \frac{1+P}{2\pi \times 10^{-8}}$$

$$\Delta\nu_D = 2\nu_0 \left(\frac{2kT\ln 2}{Mc^2}\right)^{1/2} = \frac{2}{\lambda}\left(\frac{2kT\ln 2}{M}\right)^{1/2}$$

$$M \text{ (argon)} = 40 \times 1.67 \times 10^{-27} \text{ kg}$$

Take $\lambda = 488$ nm

$$\Delta\nu_D = \frac{2}{488 \times 10^{-9}}\left(\frac{2 \times 1.38 \times 10^{-23} \times 2000 \times \ln 2}{40 \times 1.67 \times 10^{-27}}\right)^{1/2}$$

$$= 3.102 \times 10^{9} \text{ Hz}$$

Therefore when collisional broadening = Doppler broadening

$$\frac{1+P}{2\pi \times 10^{-8}} = 3.102 \times 10^{9} \quad \Rightarrow P = 193.9 \text{ atmosphere}$$

(2.8) $\chi''(\nu)$ and $\chi'(\nu)$ vary as in figure (2.16)

The slowest phase velocity will occur where $\chi'(\nu)$ is a maximum, ie at $\nu = \nu_0 - \Delta\nu/2$

Since $\chi'(\nu) = \dfrac{2(\nu_0-\nu)}{\Delta\nu}\chi''(\nu)$

$$\chi'_{max} = \chi''\left(\nu_0 - \frac{\Delta\nu}{2}\right)$$

$$\chi''\left(\nu_0 - \frac{\Delta\nu}{2}\right) = \frac{1}{2}\chi''(\nu_0)$$

Furthermore $\chi''(\nu_0) = -\dfrac{n^2\gamma(\nu_0)}{k}$

so $\chi'_{max} = \dfrac{n^2\gamma(\nu_0)}{2k}$

Take $n=1$, $\gamma(\nu_0) = 1\ m^{-1}$ $k = 2\pi/\lambda_0$

$$\chi'_{max} = \frac{300\times10^{-9}}{4\pi} \Rightarrow \Delta k = \frac{k\chi'}{2n^2} = \frac{k}{2}\left(\frac{300\times10^{-9}}{4\pi}\right)$$

From Eq. (2.131) $c' = \dfrac{c}{1+\frac{\Delta k}{k}} = c\left(1-\dfrac{\Delta k}{k}\right)$

$$\Delta c = c - c' = \frac{c\Delta k}{k} = \frac{1}{2}\,\frac{2.998\times10^{8}\times300\times10^{-9}}{4\pi}$$

$\Delta c = 3.58$ m/s

The wave slows up by 3.58 m/s

(2.9) $\Delta\nu = 10^9$ Hz

$N_2 = 5\times10^{16}\ m^{-3}$ $\qquad g_2 = g_1$

$N_1 = 5\times10^{15}\ m^{-3}$ $\qquad A_{21} = 10^8$

$\lambda_0 = 1\ \mu m$ $\qquad \tau_2 = 5\times10^{-9}$, $T_1 = 10^{-9}$

(i) $$\phi = A_{21}\tau_2\left[1 + (1 - A_{21}\tau_2)\frac{\tau_1}{\tau_2}\right]$$

$$= 10^8\times5\times10^{-9}\left[1 + (1 - 10^8 . 5\times10^{-9})\frac{1}{5}\right] = 0.55$$

$$I_s(\nu_0) = \frac{8\pi h\nu^3}{c^2\phi\, g(\nu_0,\nu)} = \frac{8\pi h\nu^3 . \pi\Delta\nu}{2c^2\phi}$$

$$= \frac{8\pi\times 6.626\times10^{-34}\times2.997\times10^8\times\pi\times10^9}{2\times10^{-18}\times0.55}$$

$$= 1.4254\times10^4\ W m^{-2}$$

(ii) $$\gamma(\nu) = (N_2 - N_1)\frac{c^2 A_{21}}{8\pi\nu^2}\, g(\nu_0,\nu)$$

$$= \frac{4.5\times10^{16}\times10^{-12}\times10^8}{8\pi}\, g(\nu_0,\nu)$$

$$= 1.79 . 10^{11} g(\nu_0,\nu) \quad m^{-1}$$

$$\chi'' = \frac{-n^2\gamma}{k} \qquad \text{Take } n = 1$$

$$\chi'' = -\frac{1.79 . 10^{11} g(\nu_0,\nu)\,\lambda_0}{2\pi} = \frac{1.79\times10^5}{2\pi} g(\nu_0,\nu)$$

$$\chi' = \frac{2(\nu_0 - \nu)}{\Delta\nu} \chi''$$

(iii) At 1 FWHM from line centre

$$g(\nu_0, \nu) = \frac{2}{5\pi \Delta\nu}$$

Therefore $\chi'' = \frac{1.79 \times 10^5}{2\pi} \cdot \frac{2}{5\pi \cdot 10^9} = 3.63 \times 10^{-6}$

Therefore $\chi' = 2\chi'' = 7.255 \times 10^{-6}$

$$\Delta k = \frac{2\pi}{\lambda} \frac{\chi'}{2} = \frac{\pi \times 7.255 \times 10^{-6}}{10^{-6}} = 22.8 \text{ m}^{-1}$$

New phase velocity is $\frac{\omega}{(k + \Delta k)} = V_p = \frac{\omega}{k\left(1 + \frac{\Delta k}{k}\right)} = \frac{\omega}{k}\left(1 - \frac{\Delta}{k}\right.$

Therefore $\Delta(\text{phase velocity}) \approx -\frac{\omega}{k} \cdot \frac{\Delta k}{k} = -c \cdot \frac{\Delta k}{k}$

$$= -c \cdot \frac{22.8}{2\pi} \cdot 10^{-6}$$

$$= \underline{4.456 \cdot 10^{-7}\, c}$$

(2.9) continued

APPROXIMATE SOLUTION

(iv) If $\tau_1 = 1\,\mu s$ and $\tau_2 = 5\,ns$ the lower level will fill up by transitions from the upper and the population inversion will be transient.

Neglecting stimulated emission $\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2}$

$$\frac{dN_2}{dt} + \frac{N_2}{\tau_2} = R_2 \longrightarrow N_2 e^{t/\tau_2} = \int R_2 e^{t/\tau_2} dt$$

$$= R_2 \tau_2 e^{t/\tau_2} + \text{constant}$$

$N_2 = 0$ at $t = 0$. Therefore $N_2 = R_2 \tau_2 (1 - e^{-t/\tau_2})$

For small t $N_2 = R_2 t$

$$\frac{dN_1}{dt} = N_2 A_{21} - \frac{N_1}{\tau_1}$$

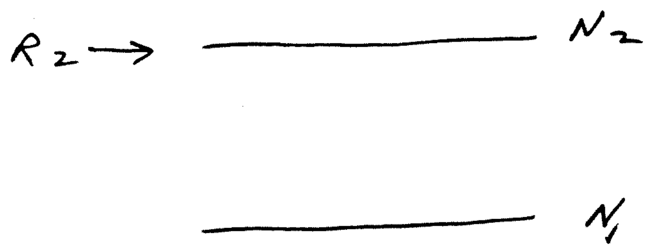
For small t $\frac{dN_1}{dt} = N_2 A_{21} = R_2 A_{21} t$

Therefore for small t $N_1 = \frac{1}{2} R_2 A_{21} t^2$

$$N_2 - N_1 = R_2 t - \frac{1}{2} R_2 A_{21} t^2 = R_2 t \left[1 - \frac{1}{2} A_{21} t \right]$$

inversion goes to zero when $t = \frac{2}{A_{21}} = 20\,ns.$

EXACT SOLUTION

(iv) 

Assume that at time $t=0$, $N_2 = N_1 = 0$

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} \Rightarrow \frac{dN_2}{dt} + \frac{N_2}{\tau_2} = R_2$$

Multiply each term by e^{t/τ_2}

$$\frac{dN_2}{dt} e^{t/\tau_2} + \frac{N_2}{\tau_2} e^{t/\tau_2} = R_2 e^{t/\tau_2}$$

Both sides can now be integrated

$$N_2 e^{t/\tau_2} = R_2 \tau_2 e^{t/\tau_2} + \text{constant}$$

From $N_2 = 0$ at $t=0$ $\Rightarrow$ constant $= -R_2\tau_2$

Therefore $N_2 = R_2\tau_2\left(1 - e^{-t/\tau_2}\right)$

$$\frac{dN_1}{dt} = N_2 A_{21} - \frac{N_1}{\tau_1} \Rightarrow \frac{dN_1}{dt} + \frac{N_1}{\tau_1} = R_2 A_{21}\tau_2\left(1 - e^{-t/\tau_2}\right)$$

gives $$N_1 e^{t/\tau_1} = \int R_2 A_{21}\tau_2 e^{t/\tau_1}\left(1 - e^{-t/\tau_2}\right) dt$$

$$= R_2 A_{21}\tau_2\tau_1 e^{t/\tau_1} - \frac{R_2 A_{21}\tau_2 e^{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)t}}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} + \text{con.}$$

$$N_1 = R_2 A_{21}\tau_2\tau_1 - \frac{R_2 A_{21}\tau_2 e^{-t/\tau_2}}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} + \text{const.}\, e^{-t/\tau_1}$$

From $N_1 = 0$ at $t=0$ const. $= \dfrac{R_2 A_{21}\tau_2}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} - R_2 A_{21}\tau_2\tau_1$

Therefore

$$N_1 = R_2 A_{21} \tau_2 \tau_1 \left(1 - e^{-t/\tau_1}\right) + \frac{R_2 A_{21} \tau_2}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} \left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$$

The population inversion is $N_2 - N_1$

$$N_2 - N_1 = R_2 \tau_2 \left(1 - e^{-t/\tau_2}\right) - R_2 A_{21} \tau_2 \tau_1 \left(1 - e^{-t/\tau_1}\right) - \frac{R_2 A_{21} \tau_2}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} \left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$$

If this inversion is positive then the system will oscillate with appropriate feedback. However, for longer times, since $\tau_1 > \tau_2$ the inversion will disappear – this is a so-called "SELF-TERMINATING LASER"

For short times

$$N_2 - N_1 = R_2 t - R_2 A_{21} \tau_2 t - \frac{R_2 A_{21} \tau_2}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} \left(\frac{t}{\tau_2} - \frac{t}{\tau_1}\right)$$

$$= R_2 t, \quad \text{which is} > 0$$

For longer times, such that $e^{-t/\tau_1} \to 0$

$$N_2 - N_1 = R_2 \tau_2 \left(1 - e^{-t/\tau_2}\right) + \frac{R_2 A_{21} \tau_2}{\left(\frac{1}{\tau_1} - \frac{1}{\tau_2}\right)} e^{-t/\tau_2}$$

At very long times

$$N_2 - N_1 = R_2 \tau_2 - R_2 A_{21} \tau_2 \tau_1 - \frac{R_2 A_{21} \tau_2}{\frac{1}{\tau_1} - \frac{1}{\tau_2}}$$

As shown on the accompanying
Mathcad plot the inversion disappears
in this case at t ~ 19 ns

(2.9) (iv)

$R2 := 10^{20}$

$\tau 2 := 5 \cdot 10^{-9}$

$\tau 1 := 10^{-6}$

$A21 := 10^{8}$

$i := 1, 2 .. 1000$

$t_i := (2 \cdot i) \cdot \frac{10^{-8}}{999}$

$$N_i := R2 \cdot \tau 2 \cdot \left(1 - e^{-\frac{t_i}{\tau 2}}\right) - R2 \cdot A21 \cdot \tau 1 \cdot \tau 2 \cdot \left(1 - e^{-\frac{t_i}{\tau 1}}\right) - R2 \cdot A21 \cdot \tau 2 \cdot \frac{e^{-\frac{t_i}{\tau 1}} - e^{-\frac{t_i}{\tau 2}}}{\left(\frac{1}{\tau 1} - \frac{1}{\tau 2}\right)}$$

INVERSION VERSUS TIME