CHAPTER FOUR

Passive Optical Resonators

4

Passive Optical Resonators

4.1 Introduction

In this chapter we shall examine the passive properties of optical resonators consisting of two plane-parallel, flat mirrors placed a distance apart. At first, the properties of standing electromagnetic waves in such a system and the way in which their stored energy is lost if the mirrors are not totally reflecting will be considered. Then we shall analyze an optical device, called the Fabry–Perot etalon or interferometer, which represents the archetypal passive resonant structure that is used in a laser. We shall see that this device has a series of equally spaced resonant frequencies and in transmission acts as a *comb* filter. The filter properties of this device allow it to be used as a high resolution instrument for analysis of the spectral content of light.

4.2 Preliminary Consideration of Optical Resonators

Before considering some of the fine detail involved in the process of laser oscillation, it will be useful to consider some preliminary aspects of the optical resonators used to provide the feedback in laser systems. Oscillation in these devices occurs because the amplifying medium is placed between suitable aligned mirrors: usually just two conormal facing mirrors are used. The passive properties of this pair of mirrors, the *optical resonator* or *cavity* of the laser, affect the way in which the oscillation

Fig. 4.1.

occurs. The resonator has resonance frequencies of its own that interact with the resonance (line center) frequency of the amplifying medium and control the output oscillation frequency of the laser. Before exploring this further, let us consider at what frequency a laser would oscillate if the resonator did not interact with the gain profile of the amplifying medium in any way. Suppose the amplifying medium has a gain profile (gain/frequency response) of a Gaussian form, as shown in Fig. (4.1). Such a gain profile occurs in a gaseous amplifying medium where the individual homogeneous lineshapes of the atoms are significantly narrower than the overall Doppler width of the spontaneous transition.

The maximum gain of the medium is at frequency ν_0, the line center, so it is perhaps logical to expect that oscillation will buildup at this frequency rather than at any other. If we view the build-up of oscillation as a process triggered by spontaneous emission we can see why this is so. A photon travelling in a direction that keeps it bouncing back and forth within the resonator is more likely to be emitted in a narrow band of frequencies $\Delta\nu_0$ near ν_0 than in some other band $\Delta\nu_1$ at frequency ν_1. As oscillation builds up, one can imagine photons spontaneously emitted at all points of the lineshape being amplified to some extent, but oscillation at ν_0 builds up fastest. As its intensity grows it depletes the atomic population by causing sufficient stimulated emission that the medium ceases to be amplifying at frequencies near ν_0 (within a few homogeneous widths, say). If the medium has a homogeneous (Lorentzian) gain profile, then since photons oscillating at ν_0 can stimulate emission from all the atoms in the medium, it is easy to see that oscillation at frequency ν_0 can suppress oscillation at any other frequency under the gain profile. The possibility of additional oscillation at frequencies far away from ν_0 in an inhomogeneously broadened gaseous amplifier is not precluded by this discussion, this in fact oftens happens, as we shall see later.

The monochromatic character of the oscillation can be predicted by a simple consideration of the shape of the gain profile of the amplifier. In the early stages of oscillation, photons with a frequency distribution $g(\nu_0, \nu)$ (the total lineshape) are being amplified in a material whose gain/frequency response is $\gamma(\nu)$ (proportional to $g(\nu_0, \nu)$). The amplification process changes the lineshape of the emitted photons circulating in the cavity by a process that is dependent on the product of $g(\nu_0, \nu)$ and $\gamma(\nu)$, that is on $[g(\nu_0, \nu)]^2$. The resulting profile of the laser radiation is dependent on higher powers of $[g(\nu_0, \nu)]^2$ as the oscillation is dependent on many passes of photons back and forth through the amplifying medium. For Gaussian lineshapes like $e^{-[2(\nu-\nu_0)/\Delta\nu_D]^2 \ell n2}$, which is like e^{-x^2/σ^2}, the product of two lineshapes produces a narrower profile, for example

$$[e^{-x^2/(\sigma^2}]^2 = e^{-2x^2/\sigma^2} = e^{-x^2/(\sigma/\sqrt{2})^2}, \tag{4.1}$$

a function that has a width $1\sqrt{2}$ less than the original. The same can also be shown to be true for Lorentzian profiles. In both cases the gain of the medium causes a narrowing of the original spontaneously emitted lineshape. Thus, we can see that in a noninteractive laser cavity, the laser oscillation will be highly monochromatic and at the line center.

Before considering the interaction that occurs when an amplifying medium is placed inside a resonator, let us review certain passive aspects of the resonator itself. Although, as we shall see later, most practical laser systems use spherical mirrors it is easier to treat the simpler plane-parallel case. The purpose of the resonator is to provide the feedback necessary to cause oscillation. The oscillation occurs as a result of spontaneous emission into those modes that keep radiation within the resonator after multiple reflections between the cavity mirrors. Emission into other modes escapes from the resonator. There are very many such modes: for example, at a wavelength of 1 μm the number of modes in 1 m^3 that lie in a frequency range of the order of a typical spontaneous emission linewidth, 10^9 Hz, (say) is, from Eq. (1.34)

$$\frac{8\pi\nu^2}{c^3}\Delta\nu = \frac{8\pi \times (3 \times 10^{14})^2 \times 10^9}{(3 \times 10)^3} \sim 8 \times 10^{13} \text{modes}.$$

Most of these modes will not lie on, or near, the normals to the resonator mirrors and will not undergo feedback. In common with a conventional electronic resonant circuit, the optical resonator has a quality factor Q that varies from one spontaneous emission mode to another. Those

Fig. 4.2.

spontaneous emission modes that lie perpendicular, or nearly so, to the parallel resonator mirror surfaces have the highest Q. The Q is defined by the relation

$$Q = \frac{2\pi\nu_0 U}{P}, \tag{4.2}$$

where ν_0 is the resonant frequency under consideration, U is the field energy stored in the resonator and P is the power dissipated by the resonator.

4.3 Calculation of the Energy Stored in an Optical Resonator

Consider the case of a standing electromagnetic wave between two perfectly conducting infinite planes of separation ℓ, as shown in Fig. (4.2). Such a wave corresponds to a mode of this resonant system that lies normal to its end planes (reflectors). Any energy stored in such a mode suffers no losses and remains stored indefinitely.

We shall calculate the electromagnetic energy stored between area A of these plates, in a volume $V = A\ell$. Take the electric field of the standing wave inside the resonator as

$$E(z,t) = E_x \sin\omega t \sin kz, \tag{4.3}$$

where in order for the electric field to be zero on each reflector, $k = n\pi/\ell$, where n is an integer[†] The total average stored energy per unit volume is

$$\frac{U}{V} = \overline{\frac{1}{2}(\epsilon E_x^2 + \mu H_y^2)}, \tag{4.4}$$

where ϵ, μ are the permittivity and permeability, respectively, of the medium filling the resonator, and the bar indicates averaging over the whole resonator. Since the magnetic field is

$$H_y = \frac{E_x}{Z} = E_x\sqrt{\frac{\epsilon}{\mu}}, \tag{4.5}$$

where Z is the impedance of the medium between the plates, the total average stored energy per unit volume is

$$\frac{U}{V} = \overline{\epsilon E_x^2}. \tag{4.6}$$

The total energy stored is

$$U = \frac{A}{T}\epsilon \int_0^\ell \int_0^T E^2(z,t)dzdt = \frac{1}{4}\epsilon E^2 V, \tag{4.7}$$

where T is the oscillation period of the field. If the power input to the resonator is P, then

$$Q = \frac{2\pi\nu_0 U}{P} = \frac{2\pi\nu_0\epsilon E^2 V}{4P}, \tag{4.8}$$

and

$$E = \sqrt{\frac{4QP}{2\pi\nu_0\epsilon V}} \tag{4.9}$$

So high electric field amplitudes are obtained, for a given power input, in a resonator with a high Q.

If we tried to build a laser with a closed resonator, for example by having an amplifying medium filling a sphere, then there would be no preferred direction in the system. Any of the $8\pi\nu^2 V d\nu/c^3$ modes of the resonator in the frequency range $d\nu$ might have similar Q factors and a confusing situation would arise, since the possibility of simultaneous oscillation on many modes having arbitrary direction and frequency would exist. This problem is avoided by using an open resonator, as then only a few of the total number of possible modes have a high Q.

† The spacing of successive standing wave values of k, π/ℓ, might appear at first glance to be in contradiction with the mode spacing $2\pi/\ell$ used earlier in our treatment of black-body radiation. However, we were counting travelling waves there and k could take both positive and negative values. If we want to count standing waves, spaced by π/ℓ, we must only include the positive octant of the **k**-space shown in Fig. (1.15).

Fig. 4.3.

4.4 Quality Factor of a Resonator in Terms of the Transmission of its End Reflectors

There is an alternative way of looking at the quality factor Q in an open resonator that has two reflectors with equal transmittance T (their fractional intensity, or energy transmission). To simplify the analysis we will assume that $T << 1$.

Consider the decay of stored energy in such a resonator of length ℓ. The initial stored energy is U_0, but at a later time this has been reduced to $U(t)$ because of transmission through the end reflectors. At any given time, equal amounts of this energy are travelling in both directions within the resonator, as shown in Fig. (4.3). In a short time dt, the energy lost from the resonator is

$$-dU = U(t)\left(\frac{Tc}{\ell}\right)dt, \tag{4.10}$$

where c is the velocity of light in the medium between the mirrors. So,

$$-\frac{dU}{U} = \left(\frac{c}{\ell}T\right)dt; \tag{4.11}$$

and with the boundary condition at $t = 0,\ U(0) = U_0$:

$$U = U_0 e^{-(cT/\ell)T}. \tag{4.12}$$

The energy stored in the resonator decays exponentially with a time constant $\tau_0 = \ell/cT$. Highly reflective ends on the resonator imply a long time constant τ_0.

The rate at which energy is dissipated in the resonator is †

$$-\frac{dU}{dt} = \left(\frac{cT}{\ell}\right) U_0 e^{-(cT/\ell)t}. \tag{4.13}$$

The Q of the resonator is

$$Q = \frac{2\pi\nu_0 \times \text{stored energy}}{\text{rate at which energy is dissipated}} = \frac{2\pi\nu_0 U_0 e^{-(cT/\ell)t}}{(cT/\ell)U_0 e^{-(cT/\ell)t}}, \tag{4.14}$$

giving

$$Q = \frac{2\pi\nu_0\ell}{cT} = 2\pi\nu_0\tau_0. \tag{4.15}$$

So a long time constant for decay of energy stored in the resonator implies a high quality factor. If the resonator contains an amplifying medium, if the gain of this medium is great enough, it prevents the decay of energy and sustains an oscillation.

4.5 Fabry–Perot Etalons and Interferometers

In the nineteenth century the French physicists Fabry and Perot[4.1] developed and analyzed an optical instrument that, although it took a slightly different form, is essentially identical to the structure that serves as the resonator in most laser systems. In the Fabry–Perot *etalon*, which consists of a pair of plane, parallel, optical interfaces or reflectors of constant separation, interference occurs between the beams of light that are multiply reflected between the two interfaces or reflectors. If the optical spacing of these interfaces or reflectors can be changed, the device is called a Fabry–Perot interferometer.

In an idealized device the plane-parallel interfaces are considered to be of infinite extent. We shall see later that in practical devices that have finite interfaces or reflectors, diffraction effects occur that lead to the loss of energy sideways and not just through the interfaces. The simplest kind of etalon consists of just a piece of plane, parallel-sided material of refractive index n immersed in a medium of refractive index n'. Ideally, the parallel faces of the device should be extremely flat, to 1/100 of a wavelength or less at the wavelength of operation. We consider what happens when a plane wave of frequency ν is incident upon this slab at angle of incident θ', as shown in Fig. (4.4).

† When the mirror transmittances are not equal and/or are not significantly smaller than unity, the treatment given here for the rate of loss of energy from the resonator is not correct. It can be shown that in these circumstances energy is not lost from the resonator in an exponential fashion.

Fig. 4.4.

Fig. 4.5.

The reflection coefficient (the ratio of the reflected to incident field amplitudes at the interface) is taken as ρ for waves travelling from n' to n and ρ' for waves travelling from n to n'. The transmission coefficients are correspondingly τ and τ'; ρ, ρ', τ, and τ' are most easily determined by the use of impedance techniques, discussed in detail in Chapter 14. If the amplitude of the incident electric vector is E_0, then the resultant amplitudes of the various reflected and transmitted waves are as shown in Fig. (4.4). Other types of plane-parallel etalons and interferometers can be analyzed in a similar manner: for example, the etalon might consist of a pair of plane reflectors held a fixed distance apart with the interspace filled with air or some other gas. In this case the medium between the plates has $n \simeq 1$.

The phase difference between two successive transmitted waves in Fig. (4.4) can be found by reference to Fig. (4.5). In this figure

$$CD = BC = \frac{\ell}{\cos\theta},$$

$$BD = 2\ell \ \tan\theta,$$
$$BE = BD\sin\theta' = BD\sin\theta\ \frac{n}{n'}, \tag{4.16}$$

where in the last of these three relations we have used Snell's law. The additional optical distance travelled by wave 2 over wave 1 is

$$\begin{aligned} n(BC+CD) - n'(BE) &= \frac{2n\ell}{\cos\theta} - 2n\ell\tan\theta\sin\theta \\ &= \frac{2n\ell}{\cos\theta} - \frac{2n\ell\sin^2\theta}{\cos\theta} \\ &= 2\ell n\cos\theta, \end{aligned} \tag{4.17}$$

so the phase difference δ between successive transmitted waves is

$$\delta = \frac{4\pi n\ell\cos\theta}{\lambda_0} + 2\epsilon, \tag{4.18}$$

where ϵ is the phase change (if any) that occurs on reflection and λ_0 is the wavelength of the wave *in vacuo*.

If the incident wave is of the form $E = |E_0|e^{i(\omega t - kz)}$, or, in terms of its complex amplitude, $E = E_0 e^{i\omega t}$ (where $E_0 = |E_0|e^{-ikz}$), in Fig. (4.4)

$$\begin{aligned} E_1 &= E_0 e^{-i\delta_0}\tau\tau', \\ E_2 &= E_0 e^{-i\delta_0}\tau\tau'\rho'^2 e^{-i\delta}, \\ E_3 &= E_0 e^{-i\delta_0}\tau\tau'\rho'^4 e^{-2i\delta}, \end{aligned} \tag{4.19}$$

and so on, where δ_0 is the phase difference introduced by the optical path AB.

The total complex amplitude of the transmitted beam is

$$E_t = E_0 e^{-i\delta_0}\tau\tau'(1 + \rho'^2 e^{-i\delta} + \rho'^4 e^{-2i\delta} + \ldots). \tag{4.20}$$

This is a geometric series with ratio $\rho'^2 e^{-i\delta}$, and first term 1. Its sum to n terms is

$$(E_t)_n = E_0 e^{-i\delta_0}\tau\tau' \frac{[1 - (\rho'^2)^n e^{-in\delta}]}{(1 - \rho'^2 e^{-i\delta})}; \tag{4.21}$$

and since $|\rho'| < 1$, the sum to infinity is

$$E_t = \frac{E_0 e^{-i\delta_0}\tau\tau'}{1 - \rho'^2 e^{-i\delta}}. \tag{4.22}$$

If the interfaces between the media with refractive indices n and n' are not made specially reflecting (by, for example, having reflective coatings placed on them) then in the case of normal incidence†

$$\rho = \frac{n' - n}{n' + n}, \tag{4.23}$$

† See Appendix 4.

$$\rho' = \frac{n - n'}{n + n'}, \tag{4.24}$$

so $\rho = -\rho'$ and there is a phase change on reflection from the interface if $n' > n$. Furthermore,

$$\tau = \frac{2n'}{n' + n}, \tag{4.25}$$

$$\tau' = \frac{2n}{n' + n}. \tag{4.26}$$

Note that $|\rho^2| = |\rho'|^2 = R$ which is called the *reflectance* of the interface: it relates the reflected and incident intensities since these are proportional to $|$ electric field $|^2$:

$$R = \left|\frac{E_{\text{reflected}}}{E_{\text{incident}}}\right|^2. \tag{4.27}$$

By a similar procedure as for the transmitted wave we can see that

$$\begin{aligned} E_r &= E_0\rho + E_0\tau\tau'\rho' e^{-i\delta} + E_0\tau\tau'\rho'^3 e^{-2i\delta} + ... \\ &= E_0[\rho + \tau\tau'\rho' e^{-i\delta}(1 + \rho'^2 e^{-i\delta} + \rho'^4 e^{-2i\delta} + ...)], \end{aligned} \tag{4.28}$$

Summing Eq. (4.28) to infinity gives

$$E_r = \frac{E_0(\rho - \rho\rho'^2 e^{-i\delta} + \tau\tau'\rho' e^{-i\delta})}{1 - \rho'^2 e^{-i\delta}}. \tag{4.29}$$

As far as the transmitted *intensity* through one interface is concerned we can define a transmittance T

$$\begin{aligned} T = \frac{I_{transmitted}}{I_{incident}} &= \left(\frac{|E^2_{transmitted}|}{Z_{in\ transmitted\ medium}}\right)\left(\frac{Z_{in\ incident\ medium}}{|E_{incident^2}|}\right) \\ &= \left|\begin{matrix}\text{transmission}\\ \text{coefficient}\end{matrix}\right|^2 \left(\frac{Z_{in\ incident\ medium}}{Z_{in\ transmitted\ medium}}\right). \end{aligned} \tag{4.30}$$

For the $n' \to n$ interface

$$Z' = \sqrt{\frac{\mu'\mu_0}{\epsilon'\epsilon_0}}, \tag{4.31}$$

where ϵ', μ' are the dielectric constant and relative magnetic permeability, respectively, of the material. If $\mu' = 1$, we can write $\sqrt{\epsilon'} = n'$. So

$$Z' = \sqrt{\frac{\mu_0}{\epsilon_0}\frac{1}{n'}}. \tag{4.32}$$

In normal incidence

$$T = \frac{\tau' Z'}{Z} = \left(\frac{2n'}{n' + n}\right)^2 \frac{n}{n'} = \frac{4nn'}{(n' + n)^2}. \tag{4.33}$$

For the $n \to n'$ interface at normal incidence

$$T = \frac{\tau'^2 Z}{Z'} = \frac{4nn'}{(n' + n)^2}. \tag{4.34}$$

So the intensity transmission coefficient T is independent of which way the wave travels through the interface and

$$R+T=\frac{(n-n')^2}{(n+n')^2}+\frac{4nn'}{(n'+n)^2}=1. \qquad (4.35)$$

Eq. (4.35) is not surprising because of energy conservation at the interface. Now

$$\tau\tau'=\frac{4nn'}{(n'+n)^2}=T, \qquad (4.36)$$

so in (4.22)

$$E_t=\frac{E_0Te^{-i\delta_0}}{1-Re^{-i\delta}}, \qquad (4.37)$$

and in (4.29)

$$E_r=\frac{E_0r(1-e^{-i\delta})}{1-Re^{-i\delta}}=\frac{E_0\sqrt{R}(1-e^{-i\delta})}{1-Re^{-i\delta}}. \qquad (4.38)$$

These formulae apply in general when R and T are the reflectance and transmittance at the appropriate angle of incidence of the plane-parallel reflecting faces that constitute the etalon. Although we have shown that the intensity reflectance coefficient of an interface between two media of different refractive indices is the same in both directions of wave propagation only in the special case of normal incidence, it can be shown to be generally true (see Appendix 4). The relations

$$R=\rho^2={\rho'}^2,$$
$$T=\tau\tau', \qquad (4.39)$$

are correct results independent of the angle of incidence.

A very popular form of Fabry–Perot etalon or interferometer consists of a pair of parallel, flat, transparent substrates to each of which a reflective coating has been applied, as shown in Fig. (4.6). Although strictly this is a composite device, containing more than two parallel flat interfaces, it can still be treated in the manner detailed above, where R and T are taken as the reflectance and transmittance of the reflective coating applied to each substrate. If the outer faces of the device X and X' are also taken into account, the system consists of a number of etalons in series and its analysis should be modified accordingly. However, it is common practice to coat faces X and X' with an antireflecting layer and have the substrates slightly wedge-shaped to prevent this difficulty from arising.

Strictly, and particularly if the reflecting faces of an air-spaced etalon are metal coated, the absorption of light in passing through the reflecting film should be taken into account. In any practical instrument it is

Fig. 4.6.

desirable for this absorption to be kept as small as possible. If the fractional energy absorption in passing through the reflective film is A', then

$$R + T + A' = 1; \tag{4.40}$$

if A' is kept very small we can still use $R + T = 1$.

The overall intensity reflection characteristic of the etalon is

$$\frac{I_r}{I_i} = \frac{E_r E_r^*}{E_0 E_0^*} = \frac{E_0 E_0^* R(1 - e^{-i\delta})(1 - e^{i\delta})}{E_0 E_0^* (1 - Re^{-i\delta})(1 - Re^{i\delta})}, \tag{4.41}$$

where I_i is the incident and I_r the reflected intensity. Eq. (4.41) reduces to

$$\frac{I_r}{I_i} = \frac{R[2 - \left(e^{i\delta} + e^{-i\delta}\right)]}{[1 + R^2 - R(e^{i\delta} + e^{-i\delta})]} = \frac{R\,(2 - 2\cos\delta)}{1 + R^2 - R\,(2\cos\delta)} \tag{4.42}$$

and writing $\cos\delta = \cos(\delta/2 + \delta/2) = \cos^2\delta/2 - \sin^2\delta/2 = 1 - 2\sin^2\delta/2$

$$\frac{I_r}{I_i} = \frac{4R\sin^2(\delta/2)}{(1 - R)^2 + 4R\sin^2(\delta/2)}. \tag{4.43}$$

In a similar way

$$\frac{I_t}{I_i} = \frac{(1 - R)^2}{(1 - R)^2 + 4R\sin^2(\delta/2)} = \frac{1}{1 + 4R/(1 - R)^2 \sin^2(\delta/2)}. \tag{4.44}$$

The transmission characteristics of the device are interesting. The transmittance is unity whenever $\sin(\delta/2) = 0$, i.e. when $\delta/2 = m\pi$ where m is an integer, i.e., when

$$\frac{4\pi n\ell\cos\theta}{\lambda_0} + 2\epsilon = 2m\pi. \tag{4.45}$$

If the phase change on reflection is 0 or π then for maximum transmittance

$$\ell = \frac{m\lambda_0}{2n\cos\theta} = \frac{m\lambda}{2\cos\theta}, \tag{4.46}$$

Fig. 4.7.

where λ is the wavelength in the material between the reflectors. In normal incidence $\ell = m\lambda/2$, a perhaps intuitively obvious result. The frequencies of maximum transmission satisfy

$$\nu_m = \frac{mc_0}{2n\ell\cos\theta}, \tag{4.47}$$

where c_0 is the velocity of light *in vacuo*. Adjacent frequencies at which the etalon shows maximum transmission are separated by a frequency

$$\Delta\nu = \frac{c_0}{2n\ell\cos\theta}. \tag{4.48}$$

This is called the *free spectral range* of the etalon. These frequencies of maximum transmission are equally spaced. A device that has this characteristic is a *comb filter*. If we allow for losses in the etalon we find that the peak transmission falls to

$$\frac{I_t}{I_0} = \frac{(1-R)^2 A}{(1-RA)^2}, \tag{4.49}$$

where the intensity of the wave on one pass through the etalon changes by a factor A.

If the transmission characteristics of a Fabry–Perot etalon are plotted for various values of R using Eq. (4.44) the curves shown in Fig. (4.7) are obtained. As R increases the sharpness of the transmission peaks increases. The quantity

$$F = \frac{\pi\sqrt{R}}{1-R}, \tag{4.50}$$

gives a measure of this sharpness and is called the *finesse* of the instrument.

Near a transmission maximum we can write $\delta = 2m\pi + \Delta$, where Δ

Fig. 4.8.

is a small angle. In this case Eq. (4.44) becomes

$$\frac{I_t}{I_i} = \frac{1}{1 + F^2\Delta^2/\pi^2}. \tag{4.51}$$

This is a Lorentzian function of Δ with FWHM 2π/F. Now, since

$$\delta = \frac{4\pi n\ell\nu\cos\theta}{c_0}, \tag{4.52}$$

it follows from Eq. (4.47) that

$$\Delta = \frac{4\pi n\ell}{c_0}(\nu - \nu_m)\cos\theta, \tag{4.53}$$

so the transmission peaks are Lorentzian functions of the frequency spacing from the transmission maximum ν_m, provided the angular deviation from the center of the peak remains a small angle. The frequency FWHM of these peaks is, therefore:

$$\Delta\nu_{1/2} = \frac{c_0}{(2n\ell\cos\theta)F} = \frac{\Delta_\nu}{F}, \tag{4.54}$$

where $\Delta\nu$ is the free spectral range. The higher the finesse the narrower become the transmission peaks.

4.6 Internal Field Strength

Since a laser consists essentially of a Fabry–Perot etalon with an amplifying medium between its reflectors, it is important to see how the external and internal electric fields of a wave passing through such an etalon are related. It is simplest to do this for a wave in normal incidence. Fig. (4.8) shows schematically the electric field amplitudes inside the etalon of the various singly and multiply reflected components of the input wave. For convenience these various components are shown displaced sideways from each other although in reality they all overlap.

The phase shift between components such as 1 and 3 or 2 and 4 is δ where as before

$$\delta = \frac{4\pi n\ell}{\lambda_0} + 2\epsilon.$$

Bearing in mind that the waves travelling from top to bottom are of the form $E_1 \propto e^{i(\omega t - kz_1)}$ and those travelling from bottom to top are of the form $E_2 \propto e^{i(\omega t - kz_2)}$, the total electric field amplitude between the etalon reflectors, for example at point X, is

$$E_{int} = E_0 e^{-i\delta_0}\tau + E_0 e^{-i(\delta-\delta_0-\epsilon)}\tau\rho' + E_0 e^{-i(\delta_0+\delta)}\tau{\rho'}^2 + \\ + E_0 e^{-i(2\delta-\delta_0-\epsilon)}\tau{\rho'}^3 + E_0 e^{-i(\delta_0+2\delta)}\tau{\rho'}^4 + ..., \tag{4.55}$$

where δ_0 is the phase shift corresponding to the path AX and ϵ is the phase change on reflection

$$\begin{aligned} E_{int} &= E_0\tau e^{-i\delta_0}\left(1 + \rho'^2 e^{-i\delta} + \rho'^4 e^{-2i\delta} + \dots \right. \\ &\quad \left. + \rho' e^{-i(\delta - 2\Delta_0 - \epsilon)} + \rho'^3 e^{-i(2\delta - 2\delta_0 - \epsilon)} + \dots\right) \\ &= E_0\tau e^{-i\delta_0}\left(\frac{1}{1-\rho'^2 e^{-i\delta}} + \frac{\rho' e^{-i(\delta - 2\delta_0 - \epsilon}}{1-\rho'^2 e^{-i\delta}}\right) \\ &= \frac{E_0\tau e^{-i\delta_0}\left(1 + \rho' e^{-i(\delta-2\delta_0-\epsilon)}\right)}{1 - Re^{-i\delta}}. \end{aligned} \tag{4.56}$$

The intracavity stored energy density depends on

$$E_{int}E_{int}^* = \frac{|E_0|^2\tau^2\left[1 + R + 2\rho'\cos(\delta - 2\delta_0 - \epsilon)\right]}{1 + R^2 - 2R\cos\delta}. \tag{4.57}$$

To study the variation of intracavity stored energy with phase shift δ it is simpler if we choose a location within the system where $2\delta_0 + \epsilon = 2p\pi$, where p is any positive or negative integer, or zero. In this case

$$E_{int}E_{int}^* = \frac{|E_0|^2\tau^2\left(1 + R + 2\sqrt{R}\cos\delta\right)}{(1 + R^2 - 2R\cos\delta)}. \tag{4.58}$$

The variation of stored energy with phase shift depends on

$$\frac{\partial}{\partial\delta}(E_{int}E_{int}^*) = \frac{2|E_0|^2\tau^2\sqrt{R}\left[-\sin\delta(1 + R^2 - 2R\cos\delta) - R\sin\partial\delta\right]}{(1 + R^2 - 2R\cos\delta)^2}. \tag{4.59}$$

Therefore, the turning points of $E_{int}E_{int}^*$ are determined by the condition

$$\sin\delta(1 + R^2 - 2R\cos\delta) + R\sin^2\delta = 0. \tag{4.60}$$

So, $E_{int}E_{int}^*$ is a maximum or a minimum when $\delta = m\pi$, where m is any integer.

When $\delta = (2m+1)\pi$, from Eq. (4.58)

$$E_{int}E^*_{int} = \frac{|E_0|^2\tau^2(1 + R - 2\sqrt{R})}{(1 + R)^2}, \tag{4.61}$$

whereas when $\delta = 2m\pi$.

$$E_{int}E^*_{int} = \frac{|E_0|^2\tau^2(1 + R + 2\sqrt{R})}{(1 - R)^2}. \tag{4.62}$$

Clearly, the maxima of $E_{int}E^*_{int}$, where $E_{int}E^*_{int}$ is proportional to the standing electric field energy density inside the etalon, occur when $\delta = 2m\pi$ and thus correspond to the transmission maxima of the system.

It is worth noting from (4.62) that for $\delta = 2m\pi, \epsilon = 0$, $E_{int}E^*_{int}$ appears to go to infinity as $R \to 1$. This curious result arises because of the way we dealt with Eq. (4.55). Since when $R \to 1, \tau \to 0$ it is clear from (4.55) that the internal field in this case goes to zero. No energy is transmitted through the etalon in this case even though Eq. (4.44) predicts the existence of apparent transmission maxima for $R \to 1, \delta = 2m\pi$. These infinitely narrow, finite height, transmission maxima are an artifact of our mathematical treatment since, in the limiting case when $R \to 1, \tau \to 0$, each term in the infinite series (4.20) is zero.

4.7 Fabry–Perot Interferometers as Optical Spectrum Analyzers

Since a Fabry–Perot etalon or interferometer has a transmission characteristic that is a function of frequency, it can be used to analyze the spectral output of a source of light. If we illuminate the etalon with white light, so that angle θ is a constant, as shown in Fig. (4.9) then for a given spacing ℓ the frequency distribution of transmitted light only shows large intensities for frequencies that satisfy

$$\nu_m = \frac{mc_0}{2n\ell\cos\theta}. \tag{4.63}$$

If we illuminate the etalon normally with a monochromatic source, then a signal will only be transmitted provided the frequency of the monochromatic source satisfies

$$\nu_0 = \frac{mc_0}{2n\ell} \tag{4.64}$$

for some integer m, as shown in Fig. (4.10). In the Fabry–Perot interferometer, in which one of the plates can be moved, the transmitted intensity as a function of plate separation, for illumination with

Fig. 4.9.

Fig. 4.10.

Fig. 4.11.

a monochromatic source of frequency ν_0 is also shown in Fig. (4.10). Transmission maxima occur for plate separations that satisfy

$$\ell = \frac{mc_0}{2n\nu_0 \cos\theta}. \tag{4.65}$$

Fig. 4.12.

$\Delta\ell$, the plate movement between successive maxima, is equal to

$$\frac{c_0}{2n\nu_0\cos\theta} = \frac{\lambda_0}{2n\cos\theta} = \frac{\lambda}{2} \quad \text{in normal incidence,} \tag{4.66}$$

where λ is the wavelength of the monochromatic signal *in the medium* between the interferometer plates. In normal incidence plate separations for maximum transmission are separated by half wavelength intervals. This half wavelength movement of one interferometer plate relative to the other is equivalent to introducing one whole wavelength additional path difference between successive transmitted rays–this is the constructive interference condition.

If it is not practical to construct a Fabry–Perot interferometer for the spectral analysis of a light signal, for example, when the source being analyzed is of a transient nature of sufficiently short duration that movement of one interferometer plate is not possible, it is still possible to use a Fabry–Perot etalon and utilize the angular discrimination between transmitted beams at different frequencies. For example, consider the case where an etalon is illuminated with a monochromatic point source, as shown in Fig. (4.11). The angles at which transmission maxima occur satisfy

$$\cos\theta_m = \frac{mc_0}{2n\ell\nu_m}. \tag{4.67}$$

The loci of rays of maximum transmission lie along the surfaces of cones with semivertical angles θ_m. The intersection of these cones with a screen produces a series of bright rings. These rings can, if necessary, be focused by a lens. This mode of usage is most useful if the source under study is diffuse, as illustrated in Fig. (4.12). In this case the lens brings all the transmitted intensity maxima at angle θ_m into focus in a ring in

Fig. 4.13.

Fig. 4.14.

the focal plane. If there is a bright point at the center of the focused ring pattern this implies $\theta_m = 0$, therefore

$$m\lambda_0 = 2n\ell. \tag{4.68}$$

For the next order transmission maximum

$$\cos\theta_{m-1} = (m-1)\frac{\lambda_0}{2n\ell}. \tag{4.69}$$

If θ_{m-1} is small, this gives

$$2n\ell\left(1 - \frac{\theta_{m-1}^2}{2}\right) = (m-1)\lambda_0. \tag{4.70}$$

Subtracting Eq. (4.68) from Eq. (4.70) gives

$$2n\ell\frac{\theta_{m-1}^2}{2} = \lambda_0, \tag{4.71}$$

and finally

$$\theta_{m-1} = \sqrt{\frac{\lambda_0}{n\ell}}. \tag{4.72}$$

Example: For $\lambda_0 = 500$nm, n = 1, $\ell = 100$mm, $\theta \simeq 2.24 \times 10^{-3}$ radian. If the focal length of the lens is f, the physical separation of adjacent rings near the center of the pattern is

$$\rho \simeq f\theta = f\sqrt{\frac{\lambda_0}{n\ell}}. \tag{4.73}$$

For $f = 500$ mm, $f\theta \simeq 1$ mm.

Although the finesse of a Fabry–Perot interferometer has been defined as

$$F = \frac{\pi\sqrt{R}}{1-R}, \tag{4.74}$$

and is the factor that determines the sharpness of transmission maxima, the true experimental finesse is also dependent on the surface flatness of the mirrors and on the angular spread of the beam incident on the spectrometer[4.2]–[4.6]. For finite-size end reflectors diffraction losses also occur. Diffraction occurs whenever a plane wave front is restricted spatially[†], and leads to an effective range of propagation directions in the resultant wave front and a further reduction in finesse.

If a Fabry–Perot system is to be used to study a source with two closely spaced monochromatic frequencies, as shown in Fig. (4.13), then these frequencies can be considered resolved if the transmitted maximum of one is a frequency $\Delta\nu_{1/2}$ from the other, that is the resolution of the instrument becomes $\Delta\nu_{1/2}$, where $\Delta\nu_{1/2} = \Delta\nu/F$, so a high finesse implies a high resolving power.

Consider the following three Fabry–Perot systems with mirrors of different reflectance.

For

$$R = 0.9, \quad F = \frac{\pi\sqrt{0.9}}{0.1} \simeq 30,$$

$$R = 0.95, \quad F = \frac{\pi\sqrt{0.95}}{0.05} = 61,$$

$$R = 0.99, \quad F = \frac{\pi}{0.01} = 314,$$

With finesses of this order, the phase difference between the transmission maximum and its half intensity points is indeed a small angle, as we had assumed previously. Table (4.1) shows the free spectral range of two of the above etalons and their resolving power for two different mirror spacings ℓ, under conditions of normal incidence operation at $\lambda = 500$ nm. We take $n = 1$ so $\Delta\nu = c_0/2\ell$.

[†] See Chapter 6.

Table (4.1). *Characteristics of Fabry–Perot interferometers with different reflectances and spacings.*

ℓ(mm)	$\Delta\nu$(Hz)	R	F	$\Delta\nu_{1/2}$(Hz)	$\nu/\Delta\nu_{1/2}$ = Resolving power
10	1.5×10^{10}	0.9	30	5×10^{8}	1.2×10^{6}
100	1.5×10^{9}	0.9	30	5×10^{7}	1.2×10^{7}
10	1.5×10^{10}	0.95	61	2.46×10^{8}	2.44×10^{6}
100	1.5×10^{9}	0.95	61	2.46×10^{7}	2.44×10^{7}

It is clear that we can increase the ability of the instrument to resolve closely-spaced wavelengths by increasing the separation of the mirrors and increasing their reflectance. However, this procedure also brings neighboring transmission peaks closer together.

Problems can arise with a Fabry–Perot system when a light source under study is emitting different frequencies that satisfy (or almost satisfy) the transmission maximum criterion for different integers m, m'

$$\nu_m = \frac{mc_0}{2n\ell}, \ \nu'_m = \frac{m'c_0}{2n\ell}. \tag{4.75}$$

In order to keep the instrument useful when this happens, it is often necessary to use it in series with some other wavelength selective device. For example, with a grating or prism monochromator as shown in Fig. (4.14). This technique is particularly successful if the ν_m, ν'_m are reasonably well separated in frequency. When it is necessary to study the spectral emission of any light source, conventional or laser, to extremely high precision, the use of a Fabry–Perot interferometer in this way is often the only feasible method of approach.

Some important physical applications of such interferometers include:

(1) High resolution spectroscopy, including measurements of the Zeeman effect and hyperfine structure.

(2) Lineshape studies that allow determination of local temperature and other parameters of the system including some excited state lifetimes.

(3) By measuring the width of spectral lines emitted by a plasma the charged particle density in the plasma can be measured. By scanning the plate spacing very fast it is possible to make such measurements even when the plasma is transient.

4.8 Problems

(4.1) Prove Eq. (4.49).

(4.2) A Fabry–Perot etalon is 3 mm thick and is made of glass (n=1.5). Diverging laser radiation of wavelength 510.6 nm is incident in a range of angles about the normal to the etalon. Calculate the radius of the first three rings observed on a screen 1 m away from the etalon. Calculate also the minimum amount by which the thickness of etalon must be changed to obtain a bright spot at the center of the pattern

(4.3) A Fabry–Perot etalon is illuminated with monochromatic radiation at a wavelength of 488.79 nm (*in vacuo*). The etalon has n=1.55 and is 7.4 mm thick. (i) Calculate the minimum change in temperature necessary to produce a bright spot at the center of the ring pattern. Take the coefficient of expansion of the etalon as $3\times\ 10^{-6}\ \mathrm{K}^{-1}$. (ii) What is the maximum divergence of the input light if only one ring is seen? (iii) A second monochromatic signal is present at 489.32 nm: (a) Do the transmitted rings have the same order within a spectral range? (b) What minimum equal reflectance of the plates of the etalon is needed to just resolve the two wavelengths?

(4.4) A Fabry–Perot etalon made of glass (n=1.5) of thickness 2mm is illuminated with diverging radiation of wavelength $\lambda_0 = 510.554$ nm from an extended source. A lens of focal length 0.5 m is used to focus the rings onto a screen. A circular aperture of radius 30 mm is cut in the screen concentric with the ring pattern. How many rings go through the aperture? What is the minimum change in refractive index required to get a bright spot at the center of the pattern?

(4.5) Analyze a Fabry–Perot etalon of index n that is bounded on one side with a medium of index n' and on the other by index of refraction n''. Calculate the transmittance as a function of the thickness of the medium. How is the definition of the finesse changed in this case?

(4.6) The *contrast ratio* of a Fabry–Perot interferometer is defined as

$$C = \frac{(I_t/I_0)_{max}}{(I_t/I_0)_{min}}.$$

Prove that

$$C = 1 + \frac{4F^2}{\pi^2}.$$

(4.7) For a real mirror $R + T + A' = 1$ where A' is the fractional energy absorption of the mirror. Derive the new version of Eq. (4.44) that holds in this case and give an equation for the new finesse.

References

[4.1] Max Born and Emil Wolf, *Principles of Optics Electromagnetic Theory of Propagation, Interference and Diffraction of light*, 6th Edition, Pergamon Press, Oxford, 1980.

[4.2] M. Françon, *Optical Interferometry*, Academic Press, New York, 1966.

[4.3] W.H. Steel, *Interferometry*, 2nd Edition Cambridge University Press, Cambridge, 1983.

[4.4] P. Hariharan, *Optical Interferometry*, Academic Press, Sydney, 1985.

[4.5] J.H. Moore, C.C. Davis, and M.A. Coplan, *Building Scientific Apparatus*, 2nd Edition Addison-Wesley, Redwood City, CA, 1988.