CHAPTER FIVE

Optical Resonators Containing Amplifying Media

5

Optical Resonators Containing Amplifying Media

5.1 Introduction

In this chapter we shall combine what we have learned about optical frequency amplification and the resonant, or feedback, characteristics of Fabry–Perot systems, in order to study laser oscillators. When a Fabry–Perot resonator is filled with an amplifying medium, laser oscillation will occur at specific frequencies if the gain of the medium is large enough to overcome the loss of energy through the mirrors and by other mechanisms within the laser medium. The onset of laser oscillation and the frequency, or frequencies, at which it occurs is governed by threshold amplitude and phase conditions, which will be derived. Once laser oscillation is estalished, it stabilizes at a level that depends on the saturation intensity of the amplifying medium and the reflectance of the laser mirrors. We shall conclude the chapter by investigating how these factors affect the output power that can be obtained from a laser, and how this can be optimized.

5.2 Fabry–Perot Resonator Containing an Amplifying Medium

Fig. (5.1) represents a Fabry-Perot resonator, whose interior is filled with an amplfying medium and which has plane mirrors. We consider the complex amplitudes of the waves bouncing backwards and forwards

Fig. 5.1.

normally between the resonator mirrors. These waves result from an incident beam with electric vector E_0 at the first mirror as shown in Fig. (5.1); where E_0 is the complex amplitude at some reference point. The reflection and transmission coefficients in the various directions at the mirrors are as shown in the figure. We include in the absorption coefficients A, A_1, A_2 at the two mirrors any reflection losses or scattering that send energy out of the resonator: we do not at this stage include diffraction losses, which result from the finite lateral dimensions of the mirrors or medium. If there were nothing inside the resonator then a wave propagating between the mirrors would propagate as $E_0e^{i(\omega t-kz)}$ to the right and $E_0e^{i(\omega t+kz)}$ to the left. The presence of a gain medium changes the otherwise passive propagation factor k to

$$k'(\omega) = k + \Delta k = k + \frac{k\chi'(\omega)}{2n^2} \tag{5.1}$$

and the gain coefficient $\gamma(\omega) = -k\chi''(\omega)/n^2$ causes the complex amplitude of the wave to change with distance as $e^{(\gamma/2)z}$. We allow for the possible existence of a distributed loss per pass given by an absorption coefficient α. Such absorption causes a fractional change in intensity for a single pass through the medium of $e^{-\alpha\ell}$. Such a distributed loss could, for example, arise from scattering by crystal imperfections in a laser rod. This distributed loss modifies the complex amplitude by a factor $e^{-i\alpha\ell/2}$ per pass. Therefore, the full propagation constant of the wave in the presence of both gain and loss is

$$k'(\omega) = k + k\frac{\chi'(\omega)}{2n^2} - \frac{ik\chi''^{(\omega)}}{2n^2} - \frac{i\alpha}{2} \tag{5.2}$$

and the wave propagates as $e^{i(\omega t\pm k'z)}$.

A wave travelling to the right with complex amplitude E_0 at plane

$z = 0$ in the resonator, the left hand mirror, has at plane ℓ, the right hand mirror, become

$$E = E_0 e^{i(\omega t - k\ell)} = E_0 e^{-ik\ell} e^{i\omega t} = E_0' e^{i\omega t}.$$

This wave then begins to propagate to the left as

$$E = E_0' e^{i(\omega t + kz)}.$$

At plane $-\ell$, the left hand mirror, with the right hand mirror now taken as the origin, it has become once more a wave travelling to the right

$$E = E_0 e^{ik\ell} e^{i(\omega t - k\ell)} = E_0 e^{-2ik\ell} e^{i\omega t}.$$

In this way we can write down the complex amplitudes of successive rays travelling at normal incidence between the two reflectors, as shown in Fig. (5.1).

The output beam through the right hand mirror arises from the transmission of waves travelling to the right: its total electric field amplitude is,

$$\begin{aligned} E_t &= E_0 t t_2 e^{-ik'\ell} + E_0 t t_2 r_1 r_2 e^{-3ik'\ell} + \ldots \\ &= E_0 t t_2 e^{-ik'\ell}(1 + r_1 r_2 e^{-2ik'\ell} + r_1^2 r_2^2 e^{-4ik'\ell} + \ldots) \\ &= \frac{E_0 t t_2 e^{-ik'\ell}}{1 - r_1 r_2 e^{-2ik'\ell}} \\ &= \frac{E_0 t t_2 e^{-i(k+\Delta k)\ell} e^{(\gamma-\alpha)\ell/2}}{1 - r_1 r_2 e^{-2i(k+\Delta k)\ell} e^{(\gamma-\alpha)\ell}}, \end{aligned} \tag{5.3}$$

where

$$\gamma(\nu) = \left[N_2 - \left(\frac{g_2}{g_1}\right) N_1\right] \left(\frac{c^2 A_{21}}{8\pi\nu^2}\right) g(\nu_0, \nu).$$

To make a further distinction between the characteristics of this active system bounded by two reflective interfaces we have replaced the ρ and τ coefficients of the last chapter with r and t, respectively.
The ratio of input to output intensities is

$$\left(\frac{E_t}{E_0}\right) = \frac{I_t}{I_0} = \frac{t^2 t_2^2 e^{(\gamma-\alpha)\ell}}{(1 - r_1 r_2 e^{-2i(k+\Delta k)\ell} e^{(\gamma-\alpha)\ell})(1 - r_1 r_2 e^{2i(k+\Delta k)\ell} e^{(\gamma-\alpha)\ell})}, \tag{5.4}$$

which becomes

$$\frac{I_t}{I_0} = \frac{t^2 t_2^2 e^{(\gamma-\alpha)\ell}}{1 + r_1^2 r_2^2 e^{2(\gamma-\alpha)\ell} - 2 r_1 r_2 e^{(\gamma-\alpha)\ell}[\cos 2(k+\Delta k)\ell]}. \tag{5.5}$$

In a passive resonator, which has no gain γ or loss α, $\Delta k = 0$, and if $r_1 = r_2 = R$

$$\frac{I_t}{I_0} = \frac{T^2}{1 + R^2 - 2R\cos 2k\ell}.$$

This is the same result as we had before, since $2k\ell = \delta$ (compare with Eq. (4.44).

In a resonator containing an active medium, as $\gamma - \alpha$ increases from zero, the denominator of Eq. (5.3) approaches zero and the whole expression blows up when

$$r_1 r_2 e^{-2i(k+\Delta k)\ell} e^{(\gamma-\alpha)\ell} = 1. \tag{5.6}$$

When this happens we have an infinite amplitude transmitted wave for a finite amplitude incident wave. In other words, a finite amplitude transmitted wave for zero incident wave – *oscillation*. Physically, Eq. (5.6) is the condition that must be satisfied for a wave to make a complete round trip inside the resonator and return to its starting point with the same amplitude and, apart from a multiple of 2π, the same phase.

Eq. (5.6) provides an amplitude condition for oscillation that gives an expression for the threshold gain constant, $\gamma_t(\nu)$,

$$r_1 r_2 e^{[\gamma_t(\omega)-\alpha]\ell} = 1. \tag{5.7}$$

To satisfy Eq. (5.6), $e^{-2i(k+\Delta k)\ell}$ must be real, which provides us with the phase condition

$$2[k + \Delta k(\nu)]\ell = 2\pi m, m = 1, 2, 3.... \tag{5.8}$$

The threshold gain coefficient can be written

$$\gamma_t(\nu) = \alpha - \frac{1}{\ell}\ell\mathrm{n} r_1 r_2, \tag{5.9}$$

which from the gain equation (2.68) gives the population inversion needed for oscillation

$$\left(N_2 - \frac{g_2}{g_1}N_1\right)_t = \frac{8\pi}{g(\nu_0,\nu)A_{21}\lambda^2}\left(\alpha - \frac{1}{\ell}\ell\mathrm{n} r_1 r_2\right). \tag{5.10}$$

For a homogeneously broadened transition the parametric variation of Eq. (5.10) that depends on the gain medium can be written as

$$\left(N_2 - \frac{g_2}{g_1}N_1\right)_t \propto \frac{\Delta\nu}{A_{21}\lambda^2}.$$

Whereas, for an inhomogeneously broadened transition since $\Delta\nu_D \propto 1/\lambda$

$$\left(N_2 - \frac{g_2}{g_1}N_1\right)_t \propto \frac{1}{A_{21}\lambda^3}.$$

Clearly, lower inversions are needed to achieve laser oscillation at longer wavelengths. It is much easier to build lasers that oscillate in the infrared than at visible, ultraviolet or X-wavelengths. For example, in an inhomogeneously broadened laser, a population inversion 10^6 times greater would be required for oscillation at 200 nm than at 20 μm (all other factors such as A_{21} being equal). In practice, since A_{21} factors generally increase at shorter wavelengths the difference in population inversion may not need to be as great as this.

In a resonator such as is shown in Fig. (5.1), if $R_1 = r_1^2 \simeq 1, R_2 = r_2^2 \simeq 1$ and distributed losses are small, a wave starting with intensity I inside the resonator will, after one complete round trip, have intensity $IR_1R_2e^{-2\alpha\ell}$, the change in intracavity intensity after one round trip is

$$dI = (R_1R_2e^{-2\alpha\ell} - 1)I. \tag{5.11}$$

This loss occurs in a time $dt = 2\ell/c$. So,

$$\frac{dI}{dt} = cI[R_1R_2e^{-2\alpha\ell} - 1]/2\ell. \tag{5.12}$$

This equation has the solution

$$I = I_0 \exp\{-[1 - R_1R_2e^{-\alpha\ell}]ct/2\ell\}, \tag{5.13}$$

where I_0 is the intensity at time $t = 0$. The time constant for intensity (energy) loss is

$$\tau_0 = \frac{2\ell}{c(1 - R_1R_2e^{-2\alpha\ell})}. \tag{5.14}$$

Now if $R_1R_2e^{-2\alpha\ell} \simeq 1$, with α small as we have assumed here, then

$$(1 - R_1R_2e^{-2\alpha\ell}) \simeq -\ell\mathrm{n}(R_1R_2e^{-2\alpha\ell}) = -\ell\mathrm{n}(R_1R_2) + 2\alpha\ell \tag{5.15}$$

and we get

$$\tau_0 = \frac{2\ell}{c(2\alpha\ell - \ell\mathrm{n}R_1R_2)} = \frac{1}{c[\alpha - (1/\ell)\ell\mathrm{n}r_1r_2]} \tag{5.16}$$

Thus, the threshold population inversion can be written

$$N_t = \frac{8\pi}{A_{21}\lambda^2 g(\nu)c\tau_0}. \tag{5.17}$$

Threshold Population Inversion - Numerical Example

For the 488 nm transition in the argon ion laser (discussed in Chapter 9)

$$\lambda = 488 \text{ nm}; c = 3 \times 10^8 \text{ ms}^{-1}; A_{21} \simeq 10^9 \text{ s}^{-1}; \Delta\nu_D \sim 3 \text{ GHz}.$$

Take $\ell = 1$ m, $R_1 = 100\%$, $R_2 = 90\%$ (typical values for a practical device). Since this is a gas laser internal losses are easily kept small so $\alpha \simeq 0$. In this case

$$\begin{aligned}\tau_0 &= 2\ell/c(1 - R_1R_2)\\ &= 66.67 \text{ ns.}\end{aligned}$$

For oscillation at, or near line center

$$g(\nu_0, \nu_0) = \frac{2}{\Delta\nu_D}\sqrt{\frac{\ell\mathrm{n}2}{\pi}} = \frac{0.94}{\Delta\nu_D} \sim \frac{1}{\Delta\nu_D}.$$

The threshold inversion is, from Eq. (5.17),

$$N_t = \frac{8\pi \times 3 \times 10^9}{10^9 \times (488 \times 10^{-9})^2 \times 3 \times 10^8 \times 66.67 \times 10^{-9}} = 1.58 \times 10^{13}\ \mathrm{m}^{-3}.$$

5.3 The Oscillation Frequency

To determine the frequency at which laser oscillation can occur we return to the phase condition, Eq. (5.8). This phase condition was

$$(k + \Delta k)\ell = m\pi, \tag{5.18}$$

which from Eq. (5.1) gives

$$k\ell\left[1 + \frac{\chi'(\nu)}{2n^2}\right] = m\pi. \tag{5.19}$$

Now, from Eq. (2.118)

$$\chi'(\nu) = \frac{2(\nu_0 - \nu)}{\Delta\nu}\chi''(\nu), \tag{5.20}$$

where ν_0 is the line center frequency and $\Delta\nu$ is its *homogeneous* FWHM, and

$$\gamma(\nu) = -\frac{k\chi''(\nu)}{n^2}. \tag{5.21}$$

So we must have

$$\frac{2\pi\nu\ell}{c}\left[1 - \frac{(\nu_0 - \nu)}{\Delta\nu}\frac{\gamma(\nu)}{k}\right] = m\pi, \tag{5.22}$$

and rearranging,

$$\nu\left[1 - \frac{(\nu_0 - \nu)}{\Delta\nu}\frac{\gamma(\nu)}{k}\right] = \frac{mc}{2\ell} = \nu_m, \tag{5.23}$$

where ν_m is the mth resonance of the passive laser resonator in normal incidence as calculated previously. Eq. (5.23) can be rewritten as

$$\nu = \nu_m - (\nu - \nu_0)\frac{\gamma(\nu)c}{2\pi\Delta\nu}. \tag{5.24}$$

We expect the actual oscillation frequency ν to be close to ν_m so we can write $(\nu - \nu_0) \simeq (\nu_m - \nu_0)$ and $\gamma(\nu) \simeq \gamma(\nu_m)$, to give

$$\nu = \nu_m - (\nu_m - \nu_0)\frac{\gamma(\nu_m)c}{2\pi\Delta\nu}. \tag{5.25}$$

At threshold

$$\gamma_t(\nu_m) = \alpha - \frac{1}{\ell}\ell\mathrm{n} r_1 r_2,$$

and if $\alpha \simeq 0, r_1 = r_2 = \sqrt{R}$

$$\gamma_t(\nu_m) = \frac{1 - R}{\ell}. \tag{5.26}$$

Fig. 5.2.

Now the FWHM of the passive resonances (the transmission intensity maxima of the Fabry–Perot) is

$$\Delta\nu_{1/2} = \frac{\Delta\nu_{FSR}}{F} = \frac{c(1-R)}{2\pi\ell\sqrt{R}}, \tag{5.27}$$

which with $R \simeq 1$ gives

$$\Delta\nu_{1/2} = \frac{c(1-R)}{2\pi\ell}, \tag{5.28}$$

and finally,

$$\nu = \nu_m - (\nu_m - \nu_0)\frac{\Delta\nu_{1/2}}{\Delta\nu}. \tag{5.29}$$

Thus, if ν_m coincides with the line center, oscillation occurs at the line center. If $\nu_m \neq \nu_0$, oscillation takes place near ν_m but is shifted slightly towards ν_0. This phenomena is called "mode-pulling" and is illustrated in Fig. (5.2).

5.4 Multimode Laser Oscillation

We have seen that for oscillation to occur in a laser system the gain must reach a threshold value $\gamma_t(\nu) = \alpha - (1/\ell)\ell n r_1 r_2$. For gain coefficients greater than this oscillation can occur at, or near (because of mode-pulling effects), one or more of the passive resonance frequencies of the Fabry–Perot laser cavity. The resulting oscillations of the system are called longitudinal modes. As oscillation at a particular one of these mode frequencies builds up, the growing intracavity energy density depletes the inverted population and gain saturation sets in. The reduction in gain continues until

$$\gamma(\nu) = \gamma_t(\nu) = \alpha - \frac{1}{\ell}\ell n r_1 r_2. \tag{5.30}$$

Fig. 5.3.

Further reduction of $\gamma(\nu)$ below $\gamma_t(\nu)$ does not occur, otherwise the oscillation would cease. Therefore, the gain is stabilized at the loss

$$\alpha - \frac{1}{\ell}\ell \mathrm{n} r_1 r_2.$$

Usually α, r_1, and r_2 are nearly constant over the frequency range covered by typical amplifying transitions, so over such moderate frequency ranges, 10^{11} Hz say, $\alpha - (1/\ell)\ell \mathrm{n} r_1 r_2$ as a function of frequency is a straight line parallel to the frequency axis. This line is called the *loss line*. At markedly different frequencies α, r_1, and r_2 can be expected to change: for example, a laser mirror with high reflectivity in the red region of the spectrum could have quite low reflectivity in the blue.

In a homogeneously broadened laser, because the reduction in gain caused by a monochromatic field is uniform across the whole gain profile, the clamping of the gain at $\gamma_t(\nu)$ leads to final oscillation at only one of the cavity resonance frequencies, the one where the original unsaturated gain was highest. We can show this schematically by plotting $\gamma(\nu)$ at various stages as oscillation builds up. Remember first the effect on $\gamma(\nu)$ produced by a monochromatic light signal of increasing intensity as shown in Fig. (5.3). Note that the gain profile is depressed uniformly even though the saturating signal is not at the line center, as predicted by Eq. (2.69).

In a laser, as oscillation begins, several such monochromatic fields start to build up at those cavity resonances where gain exceeds loss, as shown in Fig. (5.4). The oscillation stabilizes when the highest (small-signal) gain has been reduced to the loss line by saturation as shown in Figs. (5.5) and (5.6). Thus, in a *homogeneously* broadened laser, oscillation only occurs at *one* longitudinal mode frequency.

In an inhomogeneously broadened laser, the onset of gain saturation

Fig. 5.4.

Fig. 5.5.

Fig. 5.6.

due to a monochromatic signal only reduces the gain locally over a region which is of the order of a homogeneous width. Only particles whose velocities (or environments in a crystal) make their center emission frequencies lie within a homogeneous width of the monochromatic field can

Fig. 5.7.

Fig. 5.8.

interact strongly with it. Schematically, the effect of an increasing intensity monochromatic field on the gain profile is as shown in Fig. (5.7). A localized dip, or *hole*, in the gain profile occurs. If only one cavity resonance has a small-signal gain above the loss line then only this longitudinal mode oscillates. The stabilization of the oscillation might be expected to occur schematically as shown in Fig. (5.8). However, the situation is not quite as simple as this! Oscillation at this single longitudinal mode frequency implies waves travelling in both directions inside the laser cavity. These waves can be represented by

(a) the wave travelling to the right $\sim E_0 e^{i(\omega t - kz)}$,
(b) the wave travelling to the left $\sim E_0 e^{i(\omega t + kz)}$,

where we choose for convenience that $\omega = 2\pi\nu < 2\pi\nu_0$. Wave (a) can interact with particles whose center frequency is near ν. These particles are, as far as their Doppler shifts are concerned, moving away from an observer looking into the laser from right to left. Their center frequencies satisfy $\nu = \nu_0 - \mid v \mid \nu_0/c$, where positive atom velocities correspond to particles moving from left to right. Wave (b) which is travelling in the opposite direction (to the left) and is monitored, still at frequency $\nu(< \nu_0)$ by a second observer looking into the laser from left to right cannot interact with the same velocity group of particles as wave (a). The particles which interacted with wave (a) were moving away from the first observer and were Doppler shifted to lower frequencies so as to satisfy

$$\nu = \nu_0 - \frac{|\, v \,|}{c}\nu_0. \tag{5.31}$$

The second observer sees these particles approaching and their center frequency as

$$\nu = \nu_0 + \frac{|\, v \,|}{c}\nu_0, \tag{5.32}$$

so they cannot interact with wave (b). Wave (b) interacts with particles moving away from the second observer so that their velocity would be the solution of

$$\nu = \nu_0 - \frac{|\, v \,|}{c}\nu_0. \tag{5.33}$$

These particles would be monitored by the first observer at center frequency

$$\nu = \nu_0 + \frac{|\, v \,|}{c}\nu_0. \tag{5.34}$$

So the oscillating waves interact with two velocity groups of particles as shown in Fig. (5.9). This leads to saturation of the gain by a single laser mode in an inhomogeneously broadened laser both at the frequency of the mode ν and at a frequency $\nu_0 + (\nu_0 - \nu)$, which is equally spaced on the opposite side of the line center, as shown in Fig. (5.10). The power output of the laser (strictly the intracavity power) comes from those groups of particles that have gone into stimulated emission and left the two holes. The combined area of these two holes gives a measure of the laser power.

If the frequency of the oscillating mode is moved in towards the line center, the main hole and image hole begin to overlap. This corresponds physically to the left and right travelling waves within the laser cavity beginning to interact with the same velocity group of particles. As the oscillating mode moves in towards the line center, the holes overlap further, the combined area decreases and the laser output power falls, reaching a minimum at the line center. This phenomena is called the Lamb dip[5.1], named after Willis E. Lamb, Jr, who first predicted the effect, and is illustrated in Fig. (5.11). When the cavity resonance is at

Fig. 5.9.

Fig. 5.10.

Fig. 5.11.

the line center frequency ν_0, both travelling waves are interacting with the same group of atoms – those with near-zero directed velocity along the laser resonator axis.

Because hole-burning in gain saturation in inhomogeneously broad-

Fig. 5.12.

Fig. 5.13.

ened lasers is localized near the frequency of a cavity mode, one oscillating mode does not reduce the gain at other cavity modes, so simultaneous oscillation at several longitudinal modes is possible. If several such modes have small-signal gains above the loss line the oscillation stabilizes in the manner shown in Fig. (5.12a). The output frequency spectrum from the laser would appear as is shown in Fig. (5.12b). This simultaneous oscillation at several closely spaced frequencies ($c/2\ell$ apart) can be observed with a high resolution spectrometer – for example a scanning Fabry–Perot interferometer as shown in Fig. (5.13). The multiple modes are almost exactly $c/2\ell$ in frequency apart, but are not exactly equally spaced because of mode-pulling. This effect can be observed in the beat spectrum observed with a square-law optical detector (which means most optical detectors). Such a detector responds to the intensity, not the electric field of an incident light signal.

Fig. 5.14.

5.5 Mode-Beating

Suppose we shine the light from a two-mode laser on a square-law detector. The incident electric field is

$$E_i = \mathcal{R}(E_1 e^{i\omega t} + E_2 e^{i(\omega+\Delta\omega)t}), \tag{5.35}$$

where E_1 and E_2 are the complex amplitudes of the two modes and $\Delta\omega$ is the frequency spacing between them. Using real notation for these fields the output current i from the detector is

$$\begin{aligned} i &\propto \{|E_1|\cos[(\omega t+\phi_1) + |E_2|\cos[(\omega+\Delta\omega)t+\phi_2]\}^2 \\ &\propto |E_1|^2\cos^2(\omega t+\phi_1) + |E_2|^2\cos^2[(\omega+\Delta\omega)t+\phi_2] \\ &\quad + 2|E_1||E_2|\cos[(\omega t+\phi_1)\cos[\omega+\Delta\omega)t+\phi_2] \\ &\propto |E_1|^2\cos^2(\omega t+\phi_1) + |E_2|^2\cos^2[(\omega+\Delta\omega)t+\phi_2] \\ &\quad + |E_1||E_2|\cos[(2\omega+\Delta\omega)t+\phi_1+\phi_2] \\ &\quad + |E_1||E_2|\cos(\Delta\omega t+\phi_2-\phi_1). \end{aligned} \tag{5.36}$$

And since, for example,

$$|E_1|^2\cos^2(\omega t+\phi_1) = \frac{1}{2}|E_1|^2[1+\cos 2(\omega t+\phi_1)], \tag{5.37}$$

the output frequency spectrum of the detector appears to contain the frequencies $2\omega, 2(\omega+\Delta\omega), 2\omega+\Delta\omega$ and $\Delta\omega$. However, the first three of these frequencies are very high, particularly for light in the visible and infrared regions of the spectrum, and do not appear in the output of the detector. It is as if the high frequency terms are averaged to zero by the detector time response to give

$$i \propto \frac{|E_1|^2}{2} + \frac{|E_2|^2}{2} + |E_1||E_2|\cos(\Delta\omega t+\phi_2-\phi_1), \tag{5.38}$$

so only the difference frequency beat $\Delta\omega$ is observed. This result can be derived directly using the *analytic* signal of the incident electric field.

Fig. 5.15.

The *analytic* signal of a field which is $E(t) = \mathcal{R}(Ee^{i\omega t})$ is

$$V(t) = Ee^{i\omega t}, \tag{5.39}$$

so that $E(t) = \mathcal{R}[V(t)]$. The response of a square-law detector can be found directly from $i \propto V(t)V^*(t)$. The significance of this is discussed further in Chapter 23.

If the output from the square-law detector is analyzed with a radio-frequency spectrum analyzer (because it is in this frequency range where the difference frequencies between longitudinal laser modes are usually observed) different displays are obtained according to how many longitudinal modes of a multimode laser are simultaneously oscillating. Fig. (5.14) gives some examples. Because Eq. (5.29) is not quite exact, the beat frequencies can split as shown because of nonlinear mode-pulling. This splitting will only be observed if nonlinear mode-pulling is large and the spectrum analyzer that analyzes the output of the photodector has high resolution.

If a predominantly inhomogeneously broadened laser also has a significant amount of homogeneous broadening, the holes burnt in the gain curve can start to overlap, for example, when $\Delta\nu \stackrel{\sim}{>} c/2\ell$. If $\Delta\nu$ is large enough this causes neighboring oscillating modes to compete, and may lead to oscillation on a strong mode suppressing its weaker neighbors, as shown in Fig. (5.15). This effect has been observed in several laser systems, for example in the argon ion laser, where an increase in the strength of the oscillation can lead to the successive disappearance, first of every other mode, then two modes out of every three, and so on.

Fig. 5.16.

5.6 The Power Output of a Laser

When a laser oscillates, the intracavity field grows in amplitude until saturation reduces the gain to the loss line for each oscillating mode. What this means in practice can be best illustrated with reference to Fig. (5.16).

For an asymmetrical resonator, whose mirror reflectances are not equal, the distribution of standing wave energy within the resonator is not symmetrical. For example, in Fig. (5.16), if $R_2 > R_1$ the distribution of intracavity travelling wave intensity will be schematically as shown, and

$$\frac{I_3}{I_2} = R_2; \qquad \frac{I_1}{I_4} = R_1. \tag{5.40}$$

The left travelling wave, of intensity I_-, grows in intensity from I_3 to I_4 on a single pass. The right travelling wave, of intensity I_+, grows in intensity from I_1 to I_2 on a single pass. The total output intensity is

$$I_{out} = T_2 I_2 + T_1 I_4. \tag{5.41}$$

However, calculation of I_2 and I_4 is not straightforward in the general case. I_2 grows from I_1 through a gain process that depends in a complex way on $I_+ + I_-$ as does the growth of I_3 to I_4. We can identify at least three scenarios in which the calculation proceeds differently:

(a) A homogeneously broadened amplifier and single mode operation.
(b) An inhomogeneously broadened amplifier and single mode operation.
(c) An inhomogeneously broadened amplifier and multimode operation.

In both cases (b) and (c) the calculation of the output power becomes more complicated as the homogeneous contribution to the broadening grows more significant compared to $\Delta\nu_D$. This additional complexity arises because each oscillating mode burns both a primary and an image hole in the gain curve. The resultant distribution of overlapping holes makes the gain for each mode dependent not only on its own intensity but also on the intensity of the other simultaneously oscillating modes. The presence of distributed intracavity loss presents additional complications. We shall not attempt to deal with these complex situations here but will follow Rigrod[5.3] in dealing with a homogeneously broadened amplifier in which the primary intensity loss occurs at the mirrors. Inhomogeneously broadened systems and multimode operation have been discussed elsewhere by Smith.[5.4]

In a purely homogeneously broadened system the saturated gain in Fig. (5.16) is

$$\gamma(z) = \frac{\gamma_0}{1 + (I_+ + I_-)/I_s}. \tag{5.42}$$

Both I_- and I_+ grow according to $\gamma(z)$

$$\frac{1}{I_+}\frac{dI_+}{dz} = -\frac{1}{I_-}\frac{dI_-}{dz} = \gamma(z). \tag{5.43}$$

Consequently,

$$I_+ I_- = \text{constant} = C. \tag{5.44}$$

From Eq. (5.39)

$$I_4 I_1 = I_2 I_3 = C, \tag{5.45}$$

and therefore from Eq. (5.40)

$$I_2/I_4 = \sqrt{R_1/R_2}. \tag{5.46}$$

For the right travelling wave, using Eqs. (5.43) and (5.44) gives

$$\frac{1}{I_+}\frac{dI_+}{dz} = \frac{\gamma_0}{1 + (I_+ + C/I_+)/I_s}, \tag{5.47}$$

which can be integrated to give

$$\gamma_0 L = \ell\text{n}\left(\frac{I_2}{I_1}\right) + \frac{(I_2 - I_1)}{I_s} - \frac{C}{I_s}\left(\frac{1}{I_2} - \frac{1}{I_1}\right). \tag{5.48}$$

In a similar way, for the left travelling wave

$$\gamma_0 L = \ell\text{n}\left(\frac{I_4}{I_3}\right) + \frac{(I_4 - I_3)}{I_s} - \frac{C}{I_s}\left(\frac{1}{I_4} - \frac{1}{I_3}\right). \tag{5.49}$$

Adding Eqs. (5.48) and (5.49) and using Eqs. (5.40), (5.45) , and (5.46) gives

$$I_2 = \frac{I_s\sqrt{R_1}(\gamma_0 L + \ell\text{n}\sqrt{R_1 R_2})}{(\sqrt{R_1} + \sqrt{R_2})(1 - \sqrt{R_1 R_2})}. \tag{5.50}$$

Fig. 5.17.

From Eq. (5.46)

$$I_4 = I_2\sqrt{\frac{R_2}{R_1}}. \tag{5.51}$$

Now

$$T_1 = 1 - R_1 - A_1, \tag{5.52}$$

$$T_2 = 1 - R_2 - A_2, \tag{5.53}$$

so from Eqs. (5.41) and (5.50), if $A_1 = A_2 = A$

$$I_{out} = I_s\frac{(1 - A - \sqrt{R_1R_2})}{1 - \sqrt{R_1R_2}}(\gamma_0 L + \ell\text{n}\sqrt{R_1R_2}). \tag{5.54}$$

If one mirror is made perfectly reflecting, say $T_1 = 0,\ R_1 = 1$, then

$$I_{out} = T_2I_2 = \frac{T_2I_s[\gamma_0 L + \frac{1}{2}\ell\text{n}(1 - A_2 - T_2]}{(A_2 + T_2)}. \tag{5.55}$$

For a symmetrical resonator, defined by

$$R_1R_2 = R^2,$$

$$R = 1 - A - T, \tag{5.56}$$

the output intensity at each mirror is

$$\frac{I_{out}}{2} = \frac{I_s}{2}\frac{(1 - A - R)}{1 - R}(\gamma_0 L + \ell\text{n}R). \tag{5.57}$$

5.7 Optimum Coupling

To maximize the output intensity from the symmetrical resonator we must find the value of R such that $\partial I_{out}/\partial R = 0$ which gives

$$\frac{T_{opt}}{A} = \left(\frac{1 - A - T_{opt}}{A + T_{opt}}\right)[\gamma_0 L + \ell\text{n}(1 - A - T_{opt})]. \tag{5.58}$$

For small losses, such that $A + T_{opt} \ll 1$ Eq. (5.58) gives

$$\frac{T_{opt}}{A} = \sqrt{\frac{\gamma_0 L}{A}} - 1. \tag{5.59}$$

Fig. (5.17) shows the calculated optimum coupling for various values of the loss parameter A and the unsaturated gain in dB ($4.343\ \gamma_0 L$).

In practice, it should be pointed out, the optimum mirror transmittance in a laser system is generally determined empirically. For example, for the CW CO_2 laser, whose unsaturated gain varies roughly inversely with the tube diameter d, the optimum mirror transmittance has been determined to be $T \simeq L/500d$.

5.8 Problems

(5.1) A four-level laser is pumped into its pump band at a rate 10^{24} m^{-3} s^{-1}, the transfer efficiency to the upper level is 0.5. The lifetime of the upper laser level is 7×10^{-4} s. For the laser transition $A_{21} = 10^3$ s^{-1}, $\lambda_0 = 1\ \mu$m. The laser is homogeneously broadened with $\Delta\nu = 1$ GHz. Assume $n = 1.6$. The amplifying medium is 20 mm long. Neglect lower laser level population. (a) What is the gain at line center? (b) What minimum value of R_2 is needed to get oscillation if mirror 1 has $R_1 = 1$? Assume $\alpha_{distributed\ loss} = 0$.

(5.2) How many longitudinal modes will oscillate in an inhomogenously broadened gas laser with ℓ=1 m, $\gamma(\nu_0)$=1 m^{-1}, R_1=R_2=99%, α=0.001 m^{-1}, $\lambda_0 = 500$ nm, $\Delta\nu_D = 3$ GHz.

(5.3) A gas laser is operating simultaneously on five modes, none of which is at line center. The laser beam illuminates a square-law detector. Draw the RF beat spectrum that will be observed. Is the beat signal near $c/2\ell$ split?

(5.4) Derive an expression for the peak transmittance of a Fabry–Perot filter that has a round-trip in-cavity absorption that causes the intensity to change from I_0 to AI_0 on a round trip between the two mirrors.

(5.5) Derive the amplitude condition and the laser oscillation frequencies for a laser whose cavity is of length L and whose amplifying medium is of length ℓ, where $\ell < L$.

(5.6) A laser is exactly 1 m long and has a wavelength $\lambda_0 = 632.8$ nm. The mirrors of the laser have $R = 99\%$. The index of refraction of the amplifying medium is exactly 1.0001, $\Delta\nu = 100.000$ MHz. The laser operates on only the two modes nearest to the line center. The laser output illuminates a photodiode whose output is mixed with a 150 MHz local oscillator. What is the frequency of the lowest beat signal observed? Take the velocity of light in free space to be 2.997×10^8 m s^{-1}.

(5.7) A gas laser with $\lambda_0 = 325$ nm has $\gamma_0(\nu_0) = 0.1$ m^{-1}, and $\Delta\nu_D = 3$ GHz. The laser medium fills the space between two mirrors 500 mm apart. The refractive index of the laser medium can be assumed to be 1, $\Delta\nu_{homogeneous} = 10$ MHz. The distributed loss parameter is $\alpha = 0.01$ m^{-1}. What minimum equal mirror reflectance is needed to allow ten longitudinal modes to oscillate. What would happen if $\Delta\nu_{homogeneous} = 500$ MHz?

(5.8) A Fabry–Perot interferometer has two mirrors of reflectance R spaced by a distance ℓ and the space between its plates is filled with a material of absorption coefficient α, refractive index n. Calculate its contrast ratio $C = (I_t/I_0)_{max}/(I_t/I_0)_{min}$.

(5.9) A single mode homogeneously broadened laser system with a small signal gain at the oscillation frequency of 0.001 m^{-1} has total power output of 1 mW when operated in a symmetrical resonator with $R_1 = R_2 = 0.99$, L=0.5 m. The loss in each mirror is 10^{-4}. Calculate the saturation intensity.

(5.10) Write a computer program to solve Eq. (5.58) for the optimum mirror transmittance for arbitrary values of L, γ_0, and A. Find the value of T_{opt} for $L = 1$ m, $\gamma_0 = 1$ m^{-1}, $A = 10^{-4}$.

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