CHAPTER SIX

Laser Radiation

6

Laser Radiation

6.1 Introduction

In this chapter we shall examine some of the characteristics of laser radiation that distinguish it from ordinary light. Our discussion will include the monochromaticity and directionality of laser beams, and a preliminary discussion of their *coherence* properties. Coherence is a measure of the temporal and spatial phase relationships that exist for the fields associated with laser radiation.

The special nature of laser radiation is graphically illustrated by the ease with which the important optical phenomena of interference and diffraction are demonstrated using it. This chapter includes a brief discussion of these two phenomena with some examples of how they can be observed with lasers. Interference effects demonstrate the coherence properties of laser radiation, while diffraction effects are intimately connected with the beam-like properties that make this radiation special.

6.2 Diffraction

Diffraction of light results whenever a plane wave has its lateral extent restricted by an obstacle. By definition, a plane wave travelling in the z direction has no field variations in planes orthogonal to the z axis, so the derivatives $\partial/\partial x$, or $\partial/\partial y$ operating on any field component give zero. Clearly this condition cannot be satisfied if the wave strikes an obstacle:

Fig. 6.1.

at the edge of the obstacle the wave is obstructed and there must be variations in field amplitude in the lateral direction. In other words, the derivative operations $\partial/\partial x$, $\partial/\partial y$ do not give zero and the wave after passing the obstacle is no longer a plane wave. When the wave ceases to be a plane wave, its phase fronts are no longer planes and there is no unique propagation direction associated with the wave. The range of $\mathbf{k}$ vectors associated with the new wave gives rise to lateral variations in intensity observed behind the obstacle – the *diffraction pattern*.

We can show that at sufficiently great distances from the obstacle, or aperture, the diffraction pattern can be obtained as a Fourier transform of the amplitude distribution at the obstacle or aperture. There is a close parallel between this phenomenon, where light travels within a range of angular directions determined by the size of an obstacle, and the frequency spread associated with a waveform that is restricted in time. We saw in Chapter 2 that the frequency spectrum of a waveform is given by the Fourier transform of its waveform. It is perhaps not surprising to learn that the angular spectrum of a plane wave transmitted through an aperture is a Fourier transform over the shape and transmission of the aperture.

Consider two plane waves whose field variations are of the general form $e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}$ travelling at slightly different small angles and giving rise to a resultant field in the $z = 0$ plane, as shown in Fig. (6.1).

If the first wave has field amplitudes of the form

$$V_1 = V_0 e^{i(\omega t - k\sin\theta_1 x - k\cos\theta_1 z)} \tag{6.1}$$

and the second has

$$V_2 = V_0 e^{i(\omega t - k\sin\theta_2 x - k\cos\theta_2 z)}, \tag{6.2}$$

the resultant disturbance at $z = 0$ is

$$V_1 + V_2 = V_0 e^{i\omega t}\left(e^{-ik_1 x} + e^{-ik_2 x}\right). \tag{6.3}$$

Because θ is a small angle we can assume that the V_1 and V_2 vectors are parallel, and we have written

$$k_1 = k\sin\theta_1; k_2 = k\sin\theta_2, k = 2\pi/\lambda.$$

The assumption that the field vectors are parallel becomes more valid the further from the obstacle the resultant field amplitude is being calculated.

The resultant disturbance due to many such waves can be written as

$$V = V_0 e^{n\omega t}\sum_n e^{-nk_n x} \tag{6.4}$$

In the limit of infinitely many waves, this summation can be written as an integral over a continuous distribution of waves for which the amplitude distribution is $a(k_x)$. For example, the total amplitude of the group of waves travelling in the small range of angles corresponding to a range dk_x at k_x is $a(k_x)dk_x$. The total disturbance in the plane $z = 0$ is

$$V_0(x) = \int_{-\infty}^{\infty} a(k_x)e^{-ik_x x}dk_x. \tag{6.5}$$

The limits on the integral in Eq. (6.5) are set to $\pm\infty$. Implicit in this is the physical reality that $a(k_x) = 0$ for any $|k_x| > |\mathbf{k}|$.

Recognizing Eq. (6.5) as a Fourier transform we can write

$$a(k_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V_0(x)e^{ik_x x}dx. \tag{6.6}$$

Thus, if we know the field distribution at the aperture or obstacle, given by $V(x)$, we can calculate $a(k_x)$, which gives a measure of the contribution of various plane waves in the diffraction pattern. This is because an observer cannot distinguish between light coming from an aperture and a collection of plane waves that combine in space to give nonzero field amplitudes only in a region corresponding to the aperture.

6.3 Two Parallel Narrow Slits

As an example, let us look at the case of a pair of slits, which we will represent as a pair of δ-function sources. If the slits are at $x = \pm a$ in the plane $z = 0$ and are parallel to the y axis,

$$V_0(x) = \delta(x - a) + \delta(x + a). \tag{6.7}$$

So

$$a(k_x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\delta(x-a)+\delta(x+a)]e^{ik_x x}dx$$
$$= \frac{1}{2\pi}(e^{ik_x a}+e^{ik_x a}) = \frac{\cos k_x a}{\pi}. \quad (6.8)$$

Note that

$$|a(k_x)|^2 \propto \cos^2 k_x a = \cos^2\left(\frac{2\pi a\sin\theta}{\lambda}\right). \quad (6.9)$$

In the center of the diffraction pattern $|a(k_0)|^2 \propto 1$. Thus:

$$\frac{|a(k_x)|^2}{|a(k_0)|^2} = \cos^2\left(\frac{2\pi a\sin\theta}{\lambda}\right). \quad (6.11)$$

We can interpret $|a(k_x)|^2$ as the intensity in the diffraction pattern at angle θ relative to the intensity in the center of the pattern. The conditions under which Eq. (6.6) can be used to calculate the relative intensity in the diffraction pattern can be roughly stated as $z \gg D^2/\lambda$; $\theta^2 \ll z\lambda/D^2$ where D is the maximum lateral dimension of the obstacle or aperture. When these conditions are satisfied the diffraction pattern is referred to as a *Fraunhofer* diffraction pattern. Diffraction patterns observed too close to an obstacle or aperture for the above conditions to be satisfied are called *Fresnel* diffraction patterns[6.1]–[6.7].

6.4 Single Slit

The diffraction pattern for a single slit of width $2d$ can be calculated in the same way as above. The Fourier transform of the slit is

$$a(k_x) = \frac{1}{\Delta\pi}\int_{-d}^{d} e^{ik_x x}dx = \frac{\sin k_x d}{\pi k_x} \quad (6.12)$$

and the relative intensity at angle θ is

$$I = \frac{|a(k_x)|^2}{|a(k_0)|^2} = \frac{\sin^2 k_x d}{(k_x d)^2} = \frac{\sin^2(\frac{2\pi}{\lambda}d\sin\theta)}{(\frac{2\pi d\sin\theta}{\lambda})^2} \quad (6.13)$$

which is shown graphically in Fig. (6.2). The center of the diffraction pattern is an intensity maximum. The first minimum occurs when

$$\frac{2\pi d\sin\theta}{\lambda} = \pi, \quad (6.14)$$

which gives

$$\theta_{min} = \frac{\lambda}{2d} = \frac{\lambda}{w}, \quad (6.15)$$

where w is the width of the slit.

Fig. 6.2.

6.5 Two-Dimensional Apertures

We can generalize Eq. (6.6) for the case of a two-dimensional aperture by writing

$$a(k_x, k_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \xi(x, y) e^{i(k_x x + k_y y)} dxdy, \qquad (6.16)$$

where $\xi(x, y)$ is the aperture function, $k_x = |\mathbf{k}| \sin\theta$ and $k_y = |\mathbf{k}| \sin\phi$, where θ and ϕ are the angles the wave vector of the contributing plane waves make with the z axis when projected onto the xz and zy planes, respectively.

6.5.1 Circular Aperture

For a circular aperture of radius R illuminated normally with a coherent plane wave (so that all points in the aperture have the same phase) we can write

$$\begin{aligned} a(k_x, k_y) =& \frac{1}{4\pi^2} \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} [\cos(k_x x + k_y y) \\ & i \sin(k_x x + k_y y)] dxdy, \end{aligned} \qquad (6.17)$$

which can be written

$$\begin{aligned} a(k_x, k_y) = \frac{1}{4\pi^2} \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} & (\cos k_x x \cos k_y y - sin k_x x \sin k_y y \\ & + i \sin k_x x \cos k_y y + i \cos k_x \sin k_y y) dxdy. \end{aligned} \qquad (6.18)$$

All the terms in the integrand which contain sin( ) give zero because sine is an odd function: $\sin k_x x = -\sin -(k_x x)$. Thus:

$$a(k_x, k_y) = \frac{1}{4\pi^2} \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \cos k_x x \cos k_y y dxdy. \qquad (6.19)$$

Because the aperture is rotationally symmetric we can choose $k_y = 0$,

Fig. 6.3.

which gives

$$a(k_x, k_y) = \frac{1}{2\pi^2} \int_{-R}^{R} \sqrt{R^2 - x^2} \cos(k_x x) dx. \tag{6.20}$$

Put $x = R\cos\chi$, $\rho = k_x R$ to give

$$a(k_x, k_y) = \frac{R^2}{2\pi^2} \int_0^{\pi} \sin^2\chi \cos(\rho\cos\chi) d\chi. \tag{6.21}$$

The Bessel function of order 1 is defined by[6.8]

$$J_1(\rho) = \frac{\rho}{\pi} \int_0^{\pi} \sin^2\chi \cos(\rho\cos\chi) d\chi, \tag{6.22}$$

so

$$a(k_x, k_y) = \frac{R^2}{2\pi} \frac{J_1(\rho)}{\rho}(e). \tag{6.23}$$

The relative intensity corresponding to the direction k_x, k_y is

$$I = \frac{|a(k_x, k_y)|^2}{|a(k_0, k_0)|^2} = 4 \left[\frac{J_1(\rho)}{\rho}\right]^2, \tag{6.24}$$

where we have used the fact that $\lim_{\rho\to 0}[J_1(\rho)/\rho]^2 = 1/2$[6.8]. Remembering that $\rho = k_x R = |\mathbf{k}|R\sin\theta$:

$$I = 4\frac{J_1(2\pi R\sin\theta/\lambda)^2}{(2\pi R\sin\theta/\lambda)^2}. \tag{6.25}$$

This diffraction pattern is shown in Fig. (6.3). The central disc is called Airy's disc, after Sir G.B. Airy (1801–92), who first solved this problem. The minima occur when $J_1(\rho) = 0$, which is satisfied for $\rho = 1.220\pi$, 2.635π, 2.233π, 2.679π, 3.238π, 3.699π, etc. The first minimum satisfies $\rho = 1.220\pi$ which gives

$$\frac{2\pi R\sin\theta}{\lambda} = 1.22\pi \tag{6.26}$$

and

$$\sin\theta = \frac{1.22\lambda}{2R} = \frac{1.22\lambda}{D}, \tag{6.27}$$

where D is the diameter of the circular aperture.

For small angles the diffraction angle in Eq. (6.27) can be written as $\theta \simeq 1.22\lambda/D$. This angle provides a measure of when diffraction effects are important. Diffraction effects becomes negligible when $D >> \lambda$. The value of θ sets limits to the ability of optical systems to produce images, limits the ability of the focusing lens to focus plane waves to small spots, and represents a fundamental deviation of the performance of an optical system from that represented by geometrical optics. We will examine these points in further detail in Chapters 14, 15, and 16. A fundamental observation (see Chapter 14.6) is that the smallest focal spot that can be produced with a plane wave and lens, and also the smallest size object that can be imaged without excessive diffraction, are both about the size of the wavelength of the light used.

6.6 Laser Modes

In the last chapter we saw that when a laser oscillates it emits radiation at one or more frequencies that lie close to passive resonant frequencies of the cavity. These frequencies are called *longitudinal* modes. In our initial discussion of these modes we treated them as plane waves reflecting back and forth between two plane laser mirrors. In practice, laser mirrors are not always plane. Usually at least one of the laser mirrors will have concave spherical curvature. The use of spherical mirrors relaxes the alignment tolerance that must be maintained for adequate feedback to be achieved. Even if the laser mirrors are plane, the waves reflecting between them cannot be plane, as true plane waves can only exist if there is no lateral restriction of the wave fronts. Practical laser mirrors are of finite size so any wave reflecting from them will spread out because of diffraction. We have just shown that this diffractive spreading of a plane wave results when the plane wave is restricted laterally, for example by passing the wave through an aperture. Reflection from a finite size mirror produces equivalent effects. We can explain this phenomenon qualitatively by introducing the concept of Huygens secondary wavelets.† If a plane mirror is illuminated by a plane wave then each point on the

† Christian Huygens (1629–1695) was a Dutch astronomer who first suggested the concept of secondary wavelets.

Fig. 6.4.

Fig. 6.5.

mirror can be treated as a source of a spherical wave called a secondary wavelet. The overall reflected wave is the envelope of the sum total of secondary wavelets originating from every point on the mirror surface, as shown in Fig. (6.4). This construction shows that the reflected wave from a finite size mirror is not a plane wave.

The existence of diffraction in a laser resonator places restrictions on the minimum size of mirrors that can be used at a given wavelength λ and spacing ℓ. According to Eq. (6.27) a plane wave that reflects from a mirror of diameter d will spread out into a range of directions characterized by a half angle θ, where $\theta = \lambda/d$.

Consider the resonator shown in Fig. (6.5), whose reflectors have diameter d_1, d_2, respectively and spacing ℓ. A wave diffracting from mirror M_1 towards mirror M_2 will lose substantial energy past the edges of M_2 if the diffraction angle $\theta \leq d_2/\ell$. This gives us the Fresnel condition

$$\frac{d_2 d_2}{\lambda \ell} \geq 1. \tag{6.28}$$

Fig. 6.6.

The actual waves that reflect back and forth between the mirrors of a laser resonator are not plane waves but they do have characteristic spatial patterns of electric (and magnetic) field amplitude; they are called *transverse modes*. To be amplified effectively such modes must correspond to rays which make substantial numbers of specular reflections before being lost from the cavity. A transverse mode is a field configuration on the surface of one reflector that propagates to the other reflector and back, returning in the same pattern, apart from a complex amplitude factor that gives the total phase shift and loss of the round trip. To each of these transverse modes there corresponds a set of longitudinal modes spaced by approximately, $c/2\ell$.

A more detailed treatment of these transverse modes is given in Chapter 16; however, a few of their important properties will be briefly reviewed here.

(a) The nature of these transverse modes is a function of the reflector sizes, their radii of curvature, the presence of additional limiting apertures between the mirrors, and the resonator length.

(b) These modes are affected by the existence of spatial variations of gain in the laser medium.

(c) Only certain configurations of laser mirrors allow propagating transverse modes to exist that do not suffer substantial diffraction loss from the laser cavity, as illustrated in Fig. (6.6).

(d) The transverse modes, because they are essentially propagating beam solutions of Maxwell's equations, are analogous to the confined beam TEM modes in waveguides and are labelled accordingly. A laser mode of order m, n would be labelled TEM_{mn}.

(e) The transverse modes can have cartesian (rectangular) or polar (circular) symmetry. Cartesian symmetry usually arises when some

Fig. 6.7.

element in the laser cavity imposes a preferred direction on the direction of the electric and magnetic field vectors (for example, Brewster windows in the laser cavity). If distance along the resonator axis is measured by the coordinate z then a transverse mode with its electric field in the x direction would be a function of the form $E^x_{mn}(x,y,z)e^{i(\omega t \pm kz)}$ with a magnetic field of the form $H^y_{mn}(x,y,z)e^{i(\omega t \pm kz)}$

(f) Dependent on the radii of the resonator mirrors and their separation, the longitudinal modes associated with each transverse mode may have the same or different frequencies. Each individual longitudinal mode associated with a transverse model TEM_{mn} is labelled TEM_{mnq}. In the general case the longitudinal mode frequencies of two different TEM modes will be different, as shown in Fig. (6.7).

(g) Since the field distribution of the transverse modes is propagating in both directions inside the laser resonator, this field distribution is maintained in the output beam from the laser, and the resultant intensity pattern shows an xy spatial dependence in a plane perpendicular to the direction of propagation of the laser beam. These patterns have an intensity distribution

$$I(x,y,z) \propto [E^x_{mn}(x,y,z)]^2 \tag{6.29}$$

and are called mode patterns. Some examples of simple transverse mode patterns having cartesian symmetry are shown in Fig. (6.8). It can be seen that the number of xy nodal lines in the intensity pattern determines the designation TEM_{mn}.

Fig. 6.8.

6.7 Beam Divergence

Since the oscillating field distributions inside a laser are not plane waves, when they propagate through the mirrors as output beams they spread by diffraction.

The semivertical angle of the cone into which the output beam diverges is†

$$\theta_{beam} = \tan^{-1}\left(\frac{\lambda}{\pi w_0}\right) \approx \frac{\lambda}{\pi w_0}. \tag{6.30}$$

where λ is the wavelength of the output beam and w_0 is a parameter called the "minimum spot size" that characterizes the transverse mode. For the special case of two mirrors of equal radii in a confocal arrangement, as shown in Fig. (6.6), the value of w_0 is

$$(w_0)_{conf.} = \sqrt{\frac{\lambda \ell}{2\pi}}, \tag{6.31}$$

so

$$\theta_{beam(conf.)} = \tan^{-1}\frac{\lambda}{\pi}\sqrt{\frac{2\pi}{\lambda \ell}} = \tan^{-1}\sqrt{\frac{2\lambda}{\pi \ell}}. \tag{6.32}$$

If we take the specific example of 632.8 nm He–Ne laser with a symmetric confocal resonator 0.3 m long, w_0=0.17 mm and θ_{beam}=1.2 mrad=0.66°. This is a highly directional beam, but the beam does become wider the further it goes away from the laser. Such a beam is, however, highly useful in providing the perfect straight line reference. For this reason lasers are finding increasing use in construction – for tunneling, leveling, and surveying. Over a 100 m distance the laser beam just described would have expanded to a diameter of 230 mm. After travelling the

† For a derivation of this result see Chapter 16.

Fig. 6.9.

distance to the moon (~390,000 km) the beam would be ≃900 km in diameter.

6.8 Linewidth of Laser Radiation

A single longitudinal mode of a laser is an oscillation resulting from the interaction of a broadened gain curve with a passive resonance of the Fabry–Perot laser cavity. The frequency width of the gain curve is $\Delta\nu$, the frequency width of the passive cavity resonance is $\Delta\nu_{1/2} = \Delta\nu_{FSR}/F$. We expect the linewidth of the resulting oscillation to be narrower than either of these widths, as shown schematically in Fig. (6.9). It can be shown that the frequency width of the laser oscillation itself is[6.9]–[6.12]

$$\Delta\nu_{laser} = \frac{\pi h\nu_0(\Delta\nu_{1/2})^2}{P}\frac{N_2}{[N_2 - (g_2/g_1)N_1]_{threshold}}, \qquad (6.33)$$

where P is the output power.

Eq. (6.9) predicts very low linewidths for many lasers. For a typical He–Ne laser with 99% reflectance mirrors and a cavity 30 cm long

$$\Delta\nu_{1/2} = \frac{c(1-R)}{2\pi\ell} = 1.59 \text{ MHz}.$$

The factor $N_2/[N_2 - (g_2/g_1)N_1]_{threshold}$ is close to unity for a typical low power, say 1 mW, laser. Consequently,

$$\Delta\nu_{laser} = \frac{\pi \times 6.626 \times 10^{-34} \times 3 \times 10^8 \times 1.59^2 \times 10^{12}}{10^{-3} \times 632.8 \times 10^{-9}} = 2.5 \times 10^{-3} \text{ Hz}.$$

Such a small linewidth is never observed in practice because thermal instabilities and acoustic vibrations lead to variations in resonator length that further broaden the output radiation lineshape. The best observed minimum linewidths for highly stablized gas lasers operating in the visible region of the spectrum are around 10^3 Hz. Even if macroscopic thermal and acoustic vibrations could be eliminated from the system, a fundamental limit to the resonator length stability would be set by Brownian motion of the mirror assemblies. For example, consider two laser mirrors mounted on a rigid bar. The mean stored energy in the Brownian motion of the whole bar is $\overline{E} = kT$.

The frequency spread of the laser output that thereby results is

$$\Delta\nu_{Brownian} = \nu\sqrt{\frac{2kT}{YV}}, \tag{6.34}$$

where Y is the Young's modulus of the bar material and V is the volume of the mounting bar. Typical values of $\Delta\nu_{Brownian}$ are ~ 2 Hz.

6.9 Coherence Properties

Because of its extremely narrow output linewidth the output beam from a laser exhibits considerable *temporal coherence* (longitudinal coherence). To illustrate this concept, consider two points A and B a distance L apart in the direction of propagation of a laser beam, as shown in Fig. (6.10). If a definite and fixed phase relationship exists between the wave amplitudes at A and B, then the wave shows temporal coherence for a time c/L. The further apart A and B can be, while still maintaining a fixed phase relation with each other, the greater is the temporal coherence of the output beam. The maximum separation at which the fixed phase relationship is retained is called the *coherence length*, L_c, which is a measure of the length of the continuous uninterrupted wave trains emitted by the laser. The coherence length is related to the *coherence time* τ_c by $L_c = c\tau_c$. The coherence time itself is a direct measure of the monochromaticity of the laser, since by Fourier transformation, done in an analogous manner to the treatment of natural broadening,

$$\tau_c \simeq \frac{1}{\Delta\nu_L} \text{ and } L_c \simeq \frac{c}{\Delta\nu_L}. \tag{6.35}$$

The coherence length and time of a laser source are considerably better than a conventional monochromatic source (a spontaneously emitted line source). The greatly increased coherence can be demonstrated in a Michelson interferometer experiment, which allows interference between waves at longitudinally different positions in a wavefront to be studied, as shown in Fig. (6.11).

Fig. 6.10.

Fig. 6.11.

The operation of this instrument can be described as follows: Incident waves divide at the input beamsplitter. One part of the wave, A, takes the path $c \to a \to b \to a \to c \to$ detector. The other part of the incident wave, B, takes the path $c \to d \to e \to f \to e \to d \to c \to$ detector. We can write the electric field of wave A, at the detector, as

$$E_A = E_0 e^{i(\omega t + \phi_A)}, \tag{6.36}$$

where ϕ_A is the phase shift experienced along the path $c \to a \to b \to a \to c \to$ detector. Similarly, for wave B

$$E_B = E_0 e^{i(\omega t + \phi_B)}. \tag{6.37}$$

The signal from the detector can be written as

$$i(t) \propto |E_A + E_B|^2, \tag{6.38}$$

since the detector responds to the intensity of the light, which gives

$$i(t) \propto 2E_0^2[1 + \cos(\phi_B - \phi_A)]. \tag{6.39}$$

We expect maximum signal (corresponding to maximum observed illumination) if waves A and B are in phase, that is if $\phi_B - \phi_A = 2n\pi$.

Minimum signal results if $\phi_B - \phi_A = (2n+1)\pi$, when the two waves are out of phase. We can write

$$\phi_B - \phi_A = \frac{2\pi L}{\lambda}, \tag{6.40}$$

where L is the difference in path length for the two waves A and B. If L is altered, for example by moving mirror M_2 to the right in Fig. (6.11), we expect the detector signal to go up and down between its maximum and minimum values. However, this interference phenomenon will only be observed provided $L \lesssim L_c$. For a laser with $\Delta\nu \sim 1$ kHz this would require $L \lesssim 300$ km. In practice, it is not feasible to build laboratory interferometers with path differences as large as this, although large path differences can be obtained by incorporating a long optical fiber into one of the paths in Fig. (6.11). If the linewidth of the source is large then it is quite easy to demonstrate the disappearance of any interference effects of $L > L_c$. For example, with a "white" light source that covers the spectrum from 400–700 nm.

$$\Delta\nu = 3 \times 10^8 \left(\frac{1}{400 \times 10^{-9}} - \frac{1}{700 \times 10^{-a}} \right) = 3.2 \times 10^{14} \text{ Hz}$$

and therefore $L_c \sim 10^{-3}$ mm. In this case interference effects will only be observed by careful adjustment of the interferometer so both paths are almost equal. Interference fringes will be detected due to interference between the two parts of the split wave, A and B, if the coherence length of the incident wave is greater than the distance $2(cd + ef) - 2(ab)$.

It should be noted that the nonobservation of fringes in a Michelson interferometer does not imply that interference between waves is not occurring, merely that the continuing change in phase relationship between these waves shifts the position of intensity maxima and minima around in a time $\sim \tau_c$, so apparent uniform illumination results.

A laser also possesses *spatial* (lateral) coherence, which implies a definite fixed phase relationship between points separated by a distance L transverse to the direction of beam propagation. The transverse coherence length, which has similar physical meaning to the longitudinal coherence length, is

$$L_{tc} \sim \frac{\lambda}{\theta_{beam}} \sim \pi\omega_0 \tag{6.41}$$

for a laser source.

6.10 Interference

Fig. 6.12.

The existence of spatial coherence in a wavefront and the limit of its extent can be demonstrated in a classic Young's slits interference experiment. In this experiment a pair of thin, parallel slits or a pair of pinholes is illuminated normally with a spatially coherent monochromatic plane wave, as illustrated in Fig. (6.12). A laser beam operating in a *uniphase* mode, TEM_{00}, and expanded so that its central portion illuminates both slits or pinholes serves as an ideal source to illustrate this classic interference phenomenon, and at the same time test the spatial coherence properties of the laser beam.

The field amplitudes produced by the plane wave at the slits are equal and in phase and can be represented as

$$V = V_0 \cos(\omega t + \phi), \tag{6.42}$$

where ϕ is a phase factor.

Each slit acts as a source of secondary wavelets that propagate towards the observation screen. The resultant field amplitude at a point such as X will result from the superposition of the waves that have propagated from A_1 and A_2. The field amplitude at X from A_1 is

$$V_1 = b_1 V_0 \cos(\omega t + \phi - kR_1), \tag{6.43}$$

where

$$R_1 = A_1 X = \sqrt{(x-a)^2 + s^2}; \tag{6.44}$$

$k_1 = 2\pi/\lambda$ is the wavelength of the plane wave, and b_1 is a constant that depends on the geometry of the experiment. The field amplitude at X due to A_2 is

$$V_2 = b_2 V_0 \cos(\omega t + \phi - kR_2), \tag{6.45}$$

where

$$R_2 = A_2 X = \sqrt{(x+a)^2 + s^2}. \tag{6.46}$$

If $s \gg a, x$ then the magnitudes of V_1 and V_2 can be taken as equal, but they differ in phase. The total field at X is

$$V_x = V_1 + V_2 = bV_0[\cos(\omega t + \phi - kR_1) + \cos(\omega t + \phi - kR_2)], \quad (6.47)$$

where b is a constant. If $s \gg a, x$ then θ is a small angle and V_1, V_2 can be taken as parallel, independent of the polarization of the input wave. From (6.47) we get

$$V_x = 2bV_0 \cos\left[\omega t + \phi - \frac{k(R_1 + R_2)}{2}\right] \cos\left[\frac{k(R_2 - R_1)}{2}\right] \quad (6.48)$$

and the intensity at X is

$$I \propto V_x^2 \propto \cos^2\left\{\omega t + \phi - \left[\frac{k(R_1 + R_2)}{2}\right]\right\} \cos^2\left[\frac{k(R_2 - R_2)}{2}\right], \quad (6.49)$$

which gives

$$I \propto \frac{1}{2}\left\{1 + \cos 2\left[\omega t + \phi - \frac{k(R_1 + R_2)}{2}\right]\right\} \cos^2\left[\frac{k(R_2 - R_1)}{2}\right]. \quad (6.50)$$

Because ω is a very high frequency ($\sim 10^{15}$ Hz in the visible region) the detector does not see the term at frequency 2ω. It is as if the detector averages this term, which is as often positive as it is negative, to zero.† The observed intensity is therefore

$$I \propto \cos^2 \frac{k(R_2 - R_1)}{2}. \quad (6.51)$$

Now, if $s \gg a, x$,

$$R_2 = s\left[1 + \left(\frac{x + a}{s}\right)^2\right]\sqrt{1/2} = s\left[1 + \frac{1}{2}\left(\frac{x + a}{s}\right)^2\right], \quad (6.52)$$

$$R_1 = s\left[+\frac{1}{2}\left(\frac{x - a}{s}\right)^2\right], \quad (6.53)$$

so

$$R_2 - R_1 = 2ax/s, \quad (6.54)$$

which gives

$$I \propto \cos^2\left(\frac{kax}{s}\right). \quad (6.55)$$

We can write $\theta = x/s$ for $x \ll s$ giving

$$I \propto \cos^2(ka\theta) = \cos^2\left(\frac{2\pi a\theta}{\lambda}\right). \quad (6.56)$$

† For a further discussion of this point see Chapter 23.

Note that this is the same result as Eq. (6.10). The intensity is a maximum whenever

$$\frac{2\pi a\theta}{\lambda} = m\pi, \tag{6.57}$$

giving

$$\theta_{\text{max}} = \frac{m\lambda}{2a}, \tag{6.58}$$

whereas, for the minima

$$\frac{2\pi a\theta}{\lambda} = \frac{2m+1}{2}, \tag{6.59}$$

giving

$$\theta_{\text{min}} = \left(\frac{2m+1}{4}\right)\lambda. \tag{6.60}$$

The interference pattern on the screen appears as a series of equally spaced alternate bright and dark bands.

The appearance of these bright and dark bands does, however, depend crucially on the spatial coherence of the waves illuminating the two slits. If the slit variation were increased to beyond the lateral coherence length, so that $2a > L_{tc}$, then the fringe pattern would disappear.

The classic diffraction pattern of a circular aperture can also be easily observed by illuminating a small circular hole with the central portion of a TEM_{00} mode laser beam. The cleanest patterns can be obtained by focusing the laser beam with a lens and placing a small circular aperture (of size smaller than the focused laser beam) in the focal plane of the lens.

6.11 Problems

(6.1) A pair of narrow slits is illuminated with a monochromatic plane wave with $\lambda_0 = 488$ nm. At t a distance of 500 nm behind the slits the dark band spacing is 5 nm. What is the spacing of the slits?

(6.2) Calculate the interference pattern produced by four narrow slits, equally spaced by $2a$. Extend your analysis to N equally spaced slits. What happends as $N \to \infty$?

(6.3) Calculate the Fraunhofer diffraction pattern of a rectangular aperture of dimension $a \times b$.

(6.4) A Michelson interferometer with identical arms shows sharp interference effects when illuminated by a point source. A glass slab of thickness 10 mm, refractive index 1.6 is placed in one arm. No interference effects are then observed. What can you say about the coherence properties of the source?

(6.5) A gas laser with $\lambda_0 = 446$ nm has a resonator 0.5 m long. One of the laser mirrors is randomly vibrating with an amplitude of 10 nm. Estimate how much will this effectively broaden the linewidth of the emitted radiation?

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